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Concentration phenomena for some elliptic problems with almost critical exponent

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Abstract

This thesis investigates two problems in Nonlinear Partial Differential Equations, one involving the Bilaplacian operator and another involving the Laplacian operator.

First, we study the problem

$$\begin{cases} \Delta^2 u = u^{p+\varepsilon} + \lambda_\varepsilon u & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ \Delta u = u = 0 & \text{on } \partial B_1, \end{cases}$$

where B_1 is the unit open ball in $\mathbb{R}^N$, $N > 8$, $p = \frac{N+4}{N-4}$, $\varepsilon > 0$ is small and λ_ε is a positive constant depending only of ε. Our objective is to find positive solutions to the problem in the form of a tower of bubbles that concentrate at the origin as $\varepsilon \to 0$.

Secondly, we study the problem

$$\begin{cases} -\Delta u = |x|^\alpha u^{p_\alpha^* + \varepsilon} + \lambda_\varepsilon |x|^\beta u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is an open set in $\mathbb{R}^N$, $N \geq 4$, $\alpha > -2$, $-2 < \beta < N-4$, $p_\alpha^* = \frac{N+2\alpha+2}{N-2}$, $\varepsilon > 0$ is a small parameter and $\lambda_\varepsilon > 0$ depends on ε such that $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. For this problem, we obtain two results. To obtain the first result, we consider Ω to be a bounded domain in $\mathbb{R}^N$ which is symmetric with respect to $x_1, x_2, \ldots, x_N$ and contains the origin, and we find solutions in the form of a tower of bubbles of order α that concentrate at the origin as $\varepsilon \to 0$. To obtain the second result, we consider $\Omega = \mathbb{R}^N \setminus B_1$, an exterior domain, and we find solutions in the form of a tower of bubbles of order α that flatten out as $\varepsilon \to 0$.

To prove the above results, Emden-Fowler type change of variables, Lyapunov-Schmidt reduction method and Topological Degree Theory are used.

Keywords: Lyapunov-Schmidt reduction, Emden-Fowler change of variables, Topological Degree, Tower of bubbles, Critical exponent.

Resumen

Esta tesis investiga dos problemas en Ecuaciones Diferenciales Parciales No lineales, uno que involucra al operador Bilaplaciano y otro que involucra al operador Laplaciano.

Primero, estudiamos el problema

$$\begin{cases} \Delta^2 u = u^{p+\varepsilon} + \lambda_\varepsilon u & \text{en } B_1, \\ \quad u > 0 & \text{en } B_1, \\ \Delta u = u = 0 & \text{sobre } \partial B_1, \end{cases}$$

donde B_1 es la bola abierta unitaria en $\mathbb{R}^N$, $N > 8$, $p = \frac{N+4}{N-4}$, $\varepsilon > 0$ es pequeño y λ_ε es una constante positiva que depende únicamente de ε. Nuestro objetivo es encontrar soluciones al problema en forma de una torre de burbujas que se concentran en el origen cuando $\varepsilon \to 0$.

En segundo lugar, estudiamos el problema

$$\begin{cases} -\Delta u = |x|^\alpha u^{p_\alpha^* + \varepsilon} + \lambda_\varepsilon |x|^\beta u & \text{en } \Omega, \\ \quad u > 0 & \text{en } \Omega, \\ \quad u = 0 & \text{sobre } \partial\Omega, \end{cases}$$

donde Ω es un conjunto abierto en $\mathbb{R}^N$, $N \geq 4$, $\alpha > -2$, $-2 < \beta < N - 4$, $p_\alpha^* = \frac{N+2\alpha+2}{N-2}$, $\varepsilon > 0$ es un parámetro pequeño y $\lambda_\varepsilon > 0$ depende de ε tal que $\lambda_\varepsilon \to 0$ cuando $\varepsilon \to 0$. Para este problema obtenemos dos resultados. Para obtener el primer resultado, consideramos Ω siendo un dominio acotado en $\mathbb{R}^N$ el cual es simétrico con respecto a $x_1, x_2, \ldots, x_N$ y contiene el origen, y encontramos soluciones en forma de una torre de burbujas de orden α que se concentran en el origen cuando $\varepsilon \to 0$. Para obtener el segundo resultado, consideramos $\Omega = \mathbb{R}^N \setminus B_1$, que es un dominio exterior, y encontramos soluciones en forma de una torre de burbujas de orden α las cuales se aplanan cuando $\varepsilon \to 0$.

Para probar los resultados antes mencionados, se utilizan un cambio de variables tipo Emden-Fowler, el método de reducción de Lyapunov-Schmidt y la teoría del Grado Topológico.

Palabras claves: Reducción Lyapunov-Schmidt, Cambio de variables de Emden-Fowler, Grado Topológico, Torre de burbujas, Exponente crítico.

Contents

Chapter 1

Introduction

1.1 Motivation

Partial Differential Equations, or PDEs, are mathematical equations that involve functions and their partial derivatives with respect to multiple variables. Usually, the general form of a Partial Differential Equation can be written as

$$F\left(x_1, \dots, x_N, u, \frac{\partial u}{\partial x_1}, \dots, \frac{\partial^2 u}{\partial x_1^2}, \frac{\partial^2 u}{\partial x_1 \partial x_2}, \dots\right) = 0,$$

where $u = u(x_1, \dots, x_N)$ is the unknown function and $x \in \mathbb{R}^N$ is the variable.

Partial differential equations can be classified into various categories, being the most common classifications *order*, determined by the highest derivative that appears in the equation; and *linearity*, determined if the equation involves linear or nonlinear terms of u.

The importance of PDEs is not limited to just the math field, but into various fields of science and engineering. Indeed, they are fundamental in modeling physical phenomena, analyzing and designing systems, finances and model processes such as population dynamics and the spread of diseases.

Due to the aforementioned, Partial Differential Equations are crucial to mathematical modeling in numerous scientific and engineering disciplines. Their ability to describe complex systems and predict the behavior of various phenomena makes them indispensable in both theoretical and applied contexts. This makes the study of PDEs very interesting and rich, creating a wide range of problems. In particular, this work will focus on two problems, one involving the Laplacian operator and the other one involving the Bilaplacian operator. These differential operators will be addressed below.

1.2 The Laplace operator and some preliminary results

At the beginning of the 19th century, the French mathematician Pierre-Simon Laplace (1749–1827) introduced an important differential operator that would later be named after him: the Laplace operator, or Laplacian, denoted by Δ.

The Laplace operator has many interpretations, but one of the most commonly used states that the Laplacian of a function corresponds to the density of the flow of its gradient, which, in Euclidean space, leads to the definition

$$\Delta \equiv div(\nabla) = \sum_{i=1}^{N} \frac{\partial^2}{\partial {x_i}^2}.$$

In simple terms, it measures the rate at which a quantity (such as temperature, pressure or density) varies at a given point in space. To give a more precise understanding, it measures how a function changes in different directions (gradient) and then it tells us whether a flow (like wind or heat) is entering or leaving a particular point (divergence).

The Laplace operator is widely used in various branches of science and engineering. A few of such examples of its applications are given below.

1. **Heat flow:** In heat transfer problems, the Laplacian of temperature represents the spatial distribution of heat with the equation

   $$\Delta u(x,t) = \frac{\partial u(x,t)}{\partial t}.$$

   The Laplacian in this case tells us where heat is flowing in or out. Areas where the Laplacian is positive mean that heat is accumulating, while areas where it is negative mean that heat is being lost.

2. **Electrostatics:** In electrostatics, the Laplacian of electric potential gives the charge distribution in a given region with the equation

   $$\Delta V(x) = -\frac{\rho(x)}{\varepsilon_0}.$$

   A positive Laplacian indicates a region of positive charge density, while a negative one suggests negative charge.

3. **Fluid flow:** In fluid mechanics, the Laplacian appears in equations governing fluid flow, such as the Navier-Stokes equations. It describes the spatial variation of velocity, pressure, and other fluid properties.

4. **Image Processing:** In image processing, a discrete version of the Laplacian is applied to an image. By analyzing the Laplacian of an image, it is possible to identify edges and sharpen blurry details. Edges often have rapid changes in intensity, which the Laplacian picks up on.

5. **Quantum mechanics:** In quantum mechanics, the Laplacian operator arises in the Schrödinger equation, which describes the behavior of quantum particles. It represents the kinetic energy of particles in terms of their wave functions.

As we can see, this seemingly simple operator has a wide variety of uses, making its study, even to this day, very important. One of these studies was conducted in the early seventies on the stability problem of stationary stellar systems without the possibility of contact. After several studies by astronomers, physicists, and mathematicians, they managed to establish some sufficient stability conditions for certain problems where the distribution function depends only on the energy per unit mass of the star. The established stability conditions were two: the spherical polytropes must have isotropic velocity distribution and the polytropic exponent needs to be $p \geq \frac{3}{2}$. Mathematically, a polytrope refers to the solution of the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^p, \tag{1.2.1}$$

which is known as the Lane-Emden equation. In the original work, the equation was posed in $B_1 \subset \mathbb{R}^3$, the open unit ball in dimension 3, with $r = |x|$.

However, there was a problem: the stability of the system could not be guaranteed if one of the aforementioned conditions was not met. This motivated Hénon [23] to study the problem and publish his results in 1973. In that work, Hénon introduced the equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\boldsymbol{\theta}}{dr}\right) = -r^\eta\theta^{p+\frac{\eta}{2}}, \tag{1.2.2}$$

known as the Hénon equation, where $\eta > -2$. Then, through the application of numerical experiments, Hénon concluded that, in general, any polytropic model with an isotropic velocity distribution is stable, and furthermore, he determined that the critical value of the polytropic exponent was $n = 1/2$.

For higher dimensions, the equation (1.2.1) can be extended to the equation

$$-\Delta u = u^p \tag{1.2.3}$$

and it is called as Lane-Emben equation. Similarly, the equation (1.2.2) can also be extended to higher dimension to the following equation

$$-\Delta u = |x|^\alpha u^p \tag{1.2.4}$$

known as Hénon equation.

Both the Lane-Emden equation and the Hénon equation have been extensively studied for a long time, see for example [5, 8, 24, 35] and [22, 31, 32] respectively.

In particular, we are interested in studying a problem that arises from

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.5}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ and $p > 1$. The first significant contribution to problem (1.2.5) was done by Pohozaev [34]. In that work, it was proven that if Ω is star-shaped and $p \geq p_0^* := \frac{N+2}{N-2}$, then (1.2.5) has no solution. The exponent $p = p_0^*$ is the critical exponent for the embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$. This embedding is continuous but not compact when $p = p_0^*$, which causes the failure of the Palais-Smale condition and presents significant challenges in proving the existence of solutions. This raises a natural question: is it possible to find solutions when the exponent is $p \geq p_0^*$? To recover the existence of a positive solution when $p \geq p_0^*$, we have two main options: one is to perturb the *topology* of the domain (see [24, 5]); the other is to perturb the equation by *adding a lower-order term*. This thesis will focus on the latter option.

It is well known that, in a celebrated paper, Brézis and Nirenberg [6] studied the existence of solutions to the problem

$$\begin{cases} -\Delta u = u^{p_0^*} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.6}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N \geq 3$ and $\lambda > 0$. They proved that for $N \geq 4$ and $\lambda \in (0, \lambda_1)$, where λ_1 denotes the principal eigenvalue of the $-\Delta$ on Ω, there exists at least one positive solution. Additionally, they established the same result for $N = 3$, under the condition that Ω is a ball and $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$.

When p is slightly supercritical, Del Pino, Dolbeault and Musso [13] proved the existence of positive solutions to the problem

$$\begin{cases} -\Delta u = u^{p_0^*+\varepsilon} + \lambda u & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.2.7}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, $\lambda > 0$ is small enough, $\varepsilon > 0$ is a small parameter and $N \geq 4$. The cornerstone of this result is the relationship between the solutions of (1.2.7) with solutions of the critical Lane-Emden equation in the whole space

$$-\Delta U = U^{p_0^*} \quad \text{in } \mathbb{R}^N. \tag{1.2.8}$$

Indeed, in [8, 11] the *moving planes method* was used to prove that there exists a unique radially positive symmetric solution of (1.2.8), up to translation and rescaling; and that all its solutions in $\mathcal{D}^{1,2}(\mathbb{R}^N)$ have the form

$$U_{\mu,0}(x) := \gamma_N \left(\frac{\mu}{\mu^2 + |x|^2} \right)^{\frac{N+2}{N-2}}, \tag{1.2.9}$$

with $x \in \mathbb{R}^N$, $\gamma_N = (N(N-2))^{\frac{N-2}{4}}$ and $\mu > 0$. In the literature, the functions (1.2.9) are known as *bubbles*.

With respect to the problem (1.2.7), Del Pino, Dolbeault and Musso found solutions exhibiting a shape of a tower of bubbles by using a finite-dimensional reduction scheme. Specifically, given $\varepsilon > 0$ small enough, by considering $p = p_0^* + \varepsilon$ and $\lambda = \lambda_\varepsilon$, with λ_ε going to 0 as ε goes to 0 on a suitable rate of a power of ε, they constructed solutions to (1.2.7) of the form

$$u_\varepsilon(x) = \sum_{i=1}^{k} U_{\mu_i,0}(x) \, (1 + o(1)), \tag{1.2.10}$$

where $o(1) \to 0$ uniformly on B_1 as $\varepsilon \to 0$, $k \in \mathbb{N}$ and μ_i are positive parameters chosen appropriately. For $N = 3$, they also obtained sharp results in the same spirit, see [14].

The relationship established in [13] will be useful later for analyzing our problems, using the corresponding equations.

On the other hand, in the Hénon equation on the whole space

$$-\Delta U = |x|^{\alpha} U^{p} \quad \text{in } \mathbb{R}^N,$$

with $p > 1$, the weight $|x|^{\alpha}$ with $\alpha > -2$, appears multiplying the nonlinearity. This modifies the global homogeneity of the equation and shifts up the threshold between existence and nonexistence of solutions, changing the critical exponent from $p_0^* = \frac{N+2}{N-2}$ to $p_\alpha^* := \frac{N+2+2\alpha}{N-2}$. This gave rise to the critical Hénon equation in the whole space

$$-\Delta U = |x|^{\alpha} U^{p_\alpha^*} \quad \text{in } \mathbb{R}^N. \tag{1.2.11}$$

Similarly to the case of the Lane-Emden equation, it is known that, up to rescaling, there exists a unique radially symmetric positive solution of (1.2.11), and all positive radial solutions of (1.2.11) in $\mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N)$ have the form

$$U_{\mu,\alpha}(x) = \gamma_{N,\alpha} \left(\frac{\mu^{\frac{2+\alpha}{2}}}{\mu^{2+\alpha} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} \quad x \in \mathbb{R}^N, \tag{1.2.12}$$

where $\mu > 0$ and $\gamma_{N,\alpha} = ((N+\alpha)(N-2))^{\frac{N-2}{2(2+\alpha)}}$; see [20]. To be more specific, radial solutions in $\mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N)$ if $\alpha \geq 0$, or in $H^1_{\text{loc}}(\mathbb{R}^N) \cap L^{\infty}_{\text{loc}}(\mathbb{R}^N)$ if $-2<\alpha<0$, all have the form (1.2.12); see [22] and [18] respectively. Functions $U_{\mu,\alpha}$ are known as *bubbles of order* α, and they are the only positive radial functions in $\mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N)$ that achieve the best Hardy-Sobolev constant in the radial context for the inequality

$$\|\nabla u\|^2_{L^2(\mathbb{R}^N)} \geq S_{N,\alpha} \|u\|^2_{L^{\frac{2N+2\alpha}{N-2}}(\mathbb{R}^N, |x|^{\alpha})},$$

see [25, 22].

However, there are not only similarities but also differences between the Hénon equation and the Lane-Emden equation. Indeed, unlike the case $\alpha = 0$, the presence of the term $|x|^{\alpha}$ makes (1.2.11) difficult to study, since it does not allow to use the *moving planes method* to obtain the radial symmetry around any point in $\mathbb{R}^N$, leading to the appearance of nonradial solutions. This issue was addressed using bifurcation theory in [22].

This phenomenon drew attention to the Hénon problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.13}$$

where Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 3$, $\alpha > -2$ and $p > 1$. Ni [31] was the first to study this problem and found a solution when $\Omega = B_1$ and $p \in (1, p_\alpha^*)$. It was later proved that this solution is, in fact, unique up to dilations [32]. On the other hand, by a Pohozaev type identity, no solutions exist for (1.2.13) if $p \geq p_\alpha^*$ in star-shaped domains. This once again raises a natural question: is it possible to find solutions when the exponent is $p \geq p_\alpha^*$? The answer to this question is affirmative.

In the critical case, i.e. $p = p_\alpha^*$, there is an emblematic work due to Gladiali and Grossi [21].

In this work, they found positive solutions to the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p^*_\alpha} + \lambda |x|^{\beta} u & \text{in } \Omega, \\ \quad u > 0 & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.14}$$

where $p^*_\alpha = \frac{N+2+2\alpha}{N-2}$, Ω is a smooth and bounded domain in $\mathbb{R}^N$, $N \geq 4$, $0 \in \Omega$, $\alpha \geq 0$, $\beta \geq 0$ and $\lambda > 0$ is a positive and small parameter. It is important to note that this result managed to cover the cases where α is odd, which had not been addressed until then, making it a very significant result.

In the supercritical case, i.e. $p > p^*_\alpha$, Liu [27] studied and proved the existence of a tower of bubble type of solution for the more general problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p^*_\alpha + \varepsilon} + \lambda_\varepsilon |x|^{\beta} u & \text{in } B_1, \\ \quad u > 0 & \text{in } B_1, \\ \quad u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.2.15}$$

where B_1 is the open unit ball in $\mathbb{R}^N$, $N \geq 3$, $\alpha \geq 0$, $-2 < \beta < N-4$, $p^*_\alpha = \frac{N+2\alpha+2}{N-2}$, $\varepsilon > 0$ is a small parameter and $\lambda_\varepsilon > 0$ depends on ε such that $\lambda_\varepsilon \to 0$ as ε goes to 0. One of our objectives is to extend this result to more general domains, including the case of the exterior domain, as well as considering the range $-2 < \alpha < 0$.

1.3 The Bilaplacian operator and some preliminary results

The bilaplacian operator plays a significant role in various fields such as physics and engineering. Mathematically, it is defined in N dimensions as

$$\begin{aligned} \Delta^2 u(x) &= \Delta(\Delta u(x)) \\ &= \left(\sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} \right) \left(\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} \right) u(x). \end{aligned}$$

In practice, partial differential equations involving the bilaplacian operator are used to solve problems coming from different applications areas. Some exaples are given below.

1. **Solid mechanics:** The bilaplacian operator models the deflection of thin plates through the biharmonic equation, also known as the Kirchhoff-Love equation

   $$\Delta^2 u(x,y) = -\frac{q(x,y)}{D},$$

   where $u(x,y)$ is the deflection of the plate in the mid-plane, $q(x,y)$ is the load applied to the plate and D is the flexural rigidity of the plate. This equation is derived from the equilibrium equations and Kirchhoff's hypotheses, such as the material of the plate being linearly elastic, the thickness of the plate being smaller than the other two dimensions, the plate being initially flat and the deformations being small compared to the thickness of the plate.

2. **Fluid mechanics:** The biharmonic operator is essential for modeling the stream function of steady, incompressible, and viscous flows. It provides a means to obtain velocity and pressure fields from the stream function, which are critical for understanding and predicting fluid behavior.

3. **Mathematical physics:** The biharmonic operator appears in quantum mechanics, particularly in problems involving the Schrödinger operator.

Recent research has made significant progress in understanding positive solutions for biharmonic equations, particularly through the use of advanced techniques from nonlinear analysis and functional analysis. The fourth-order Lane-Emden equation

$$\Delta^2 u = u^p \quad \text{in } \mathbb{R}^N, \tag{1.3.1}$$

has been studied in depth and shares some results with its second-order counterpart. When $N \geq 5$ and $1 < p < \frac{N+4}{N-4}$, there are no positive solutions, see [38]; whereas if $p = \frac{N+4}{N-4}$, the distinguished paper by Lin [26] establishes that all positive solutions of the problem (1.3.1), after rescaling and translation, possess the same structure

$$U_{\mu,2,0}(x) = \overline{\gamma}_{N,2}\left(\frac{\mu}{\mu^2 + |x|^2}\right)^{\frac{N-4}{2}}, \tag{1.3.2}$$

where $\mu > 0$ and $\overline{\gamma}_{N,2} = [N(N-4)(N-2)(N+2)]^{\frac{N-4}{8}}$. Moreover, similarly to the second-order case, $U_{\mu,2,0}$ satisfies the condition to achieve the best constant for the Sobolev inequality

$$\|\Delta u\|^2_{L^2(\mathbb{R}^N)} \geq S_N \|u\|^2_{L^{\frac{2N}{N-4}}(\mathbb{R}^N)},$$

see [25].

Consider now the problem

$$\begin{cases} \Delta^2 u = & u^p & \text{in } \Omega, \\ u > & 0 & \text{in } \Omega, \\ \Delta u = u = & 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3.3}$$

where Ω is an open bounded domain in $\mathbb{R}^N$. When $\Omega = B_1$, the open unit ball in $\mathbb{R}^N$, the problem (1.3.3) satisfies the Palais-Smale condition when $1 < p < \frac{N+4}{N-4}$, allowing the use of energy functionals associated with (1.3.3) to find infinite solutions through minimizers [19]. However, when $p \geq \frac{N+4}{N-4}$, the Palais-Smale condition is not satisfied, and thus the injection $H^2 \cap H^1_0(\Omega) \hookrightarrow L^{\frac{2N}{N-4}}$ loses compactness, leading to a non-existence result for positive solutions in star-shaped domains [38]. Consequently, $p = \frac{N+4}{N-4}$ is the critical exponent for the equation (1.3.3).

Again, as in the case of the Laplacian operator, a natural question arises: it is possible to find solutions when $p \geq \frac{N+4}{N-4}$? The answer to this question is once again affirmative.

For $p = \frac{N+4}{N-4}$, by using an algebraic topological argument due to Bahri and Coron ([5]), Ebobisse and Ahmedou [17] prove how the topology of the domain influences the existence of solutions to the problem (1.3.3). Recently, a related result was obtained by Sangdon and Seunghyeok [36], who proved the existence of solutions in a punctured domain for a system of equations that, in a particular case, can be reduced to the problem (1.3.3).

For $p > \frac{N+4}{N-4}$, a recent result by Martino and Maalaoui [29] proved the existence of solutions by adding a small hole within the domain. Additionally, a more recent result by Alarcón and Pistoia [3] found solutions in an exterior domain. But is it possible to find solutions in the supercritical case in a bounded domain without altering its topology? Our work provides an answer to this question.

1.4 Presentation of the problems and results obtained

In this section, we present the problems to be studied, as well as the results obtained.

The first problem studied is

$$\begin{cases} \Delta^2 u = u^{p+\varepsilon} + \lambda_\varepsilon u & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ \Delta u = u = 0 & \text{on } \partial B_1, \end{cases} \tag{1.4.1}$$

where B_1 is the open unit ball in $\mathbb{R}^N$, $N > 8$, $p = \frac{N+4}{N-4}$, $\varepsilon > 0$ is small and λ_ε is a positive constant depending on ε.

The objective is to find positive solutions, under Navier boundary conditions, which take the form of a tower of bubble concentrating at the origin. The result found is

Theorem 1.4.1. *Let B_1 be the open unit ball in $\mathbb{R}^N$, with $N > 8$ and $k \in \mathbb{N}$. Then, there exist numbers $\overline{\eta}_k > 0$ and $\overline{\varepsilon}_k > 0$, such that if $\eta \in (\overline{\eta}_k, +\infty)$, $\varepsilon \in (0, \overline{\varepsilon}_k)$ and $\lambda_\varepsilon = \eta\varepsilon^{\frac{N-8}{N-4}}$, then there is a pair of solutions $u^\pm_{k,\varepsilon} : B_1 \to \mathbb{R}$ to problem (1.4.1) defined as*

$$u^\pm_{k,\varepsilon}(x) = \sum_{i=1}^{k} P_{B_1} U_{\mu^\pm_{i,\varepsilon}}(x) + \varphi_\varepsilon(x), \quad x \in B_1 \tag{1.4.2}$$

where $\mu^\pm_{i,\varepsilon} = e^{\frac{2}{N-4}\xi^\pm_{i,\varepsilon}} = M^\pm_i \varepsilon^{\frac{1-2i}{N-4}}$ for each $i = 1, 2, \dots, k$, with $M^\pm_1, M^\pm_2, \dots, M^\pm_k$, being positive constants that depend only on N and k; and $\varphi_\varepsilon(x)$ is a function of lower order than every $P_{B_1} U_{\mu^\pm_{i,\varepsilon}}(x)$ in B_1, with $\varphi_\varepsilon \to 0$ uniformly on compacts as $\varepsilon \to 0$.

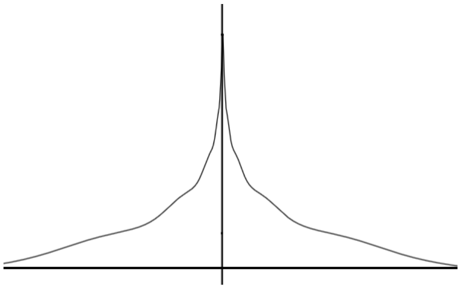

Fig. 1. Functions (1.4.2) taking the form of a tower of bubbles, choosing $k = 20$, $\varepsilon = 0.01$ and $N = 6$.

The second problem studied is

$$\begin{cases} -\Delta u = & |x|^{\alpha} u^{p^*_\alpha+\varepsilon} + \lambda_\varepsilon |x|^{\beta} u \quad \text{in } \Omega, \\ \quad u > & 0 \qquad\qquad\qquad\qquad \text{in } \Omega, \\ \quad u = & 0 \qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{1.4.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$, which is symmetric respect to $x_1, x_2, \ldots, x_N$ and contains the origin, $\alpha > -2$, $-2 < \beta < N-4$, $p^*_\alpha = \frac{N+2\alpha+2}{N-2}$, $\varepsilon > 0$ is a small parameter and $\lambda_\varepsilon > 0$ depends on ε, with $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$.

Our main focus lies in finding positive solutions that take the form of a tower of bubbles of order α, exhibiting a concentration phenomenon at the origin. The result found is

Theorem 1.4.2. *Let Ω be a bounded smooth domain in $\mathbb{R}^N$, $N \geq 4$, which is symmetric with respect to $x_1, x_2, \ldots, x_N$ and contains the origin. Let $k \in \mathbb{N}$, $\alpha > -2$ and $-2 < \beta < N-4$. In addition, suppose that one and only one of the following conditions holds: either*

H_1) α is not an even integer, or

H_2) $\alpha = 2(m-1)$ for some $m \in \mathbb{N}$, $m \geq 2$, and Ω is invariant for the group $O(M)$, where M is an integer $M > \sigma_m$.

Then, there exist numbers $\eta_k > 0$ and $\bar{\varepsilon}_k > 0$ such that if $\eta \in (\eta_k, +\infty)$, $\varepsilon \in (0, \bar{\varepsilon}_k)$ and $\lambda_\varepsilon = \eta \varepsilon^{\frac{N-4-\beta}{N-2}}$, then there are positive constants $\mu^{\pm}_{j\varepsilon,\alpha}$, for $j = 1, 2, \ldots, k$, which depend on k, N and η, and a pair of solutions $u^{\pm}_{k,\varepsilon} : \Omega \to \mathbb{R}$ to problem (1.4.3) such that

$$u^{\pm}_{k,\varepsilon}(x) = \sum_{j=1}^{k} P_\Omega U_{\mu^{\pm}_{j\varepsilon},\alpha}(x) + \varphi_\varepsilon(x), \quad x \in \Omega, \tag{1.4.4}$$

where $\mu^{\pm}_{i,\varepsilon} = M^{\pm}_i \varepsilon^{\frac{2i-1}{N-2}}$ for each $i = 1, 2, \ldots, k$, with $M^{\pm}_1, M^{\pm}_2, \ldots, M^{\pm}_k$, being positive constants that depend only on N, k and α, φ_ε is a function of lower order than every $P_\Omega U_{\mu^{\pm}_{j\varepsilon},\alpha}$ in Ω, with $\varphi_\varepsilon \to 0$ uniformly on compacts contained in Ω as $\varepsilon \to 0$.

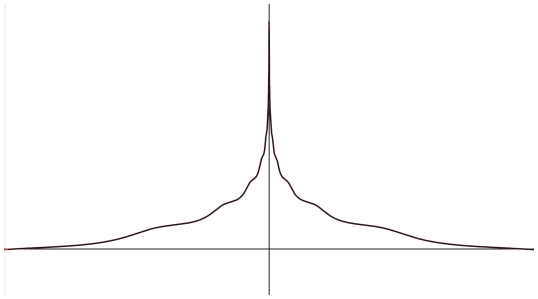

Fig. 2. Functions (1.4.4) taking the form of a tower of bubbles, choosing $k = 20$, $\varepsilon = 0.02$, $\alpha = 10$ and $N = 6$.

Remark 1.4.3. *After carrying out a dominated balance on the equation, it can be directly verified that the announced solution is valid for $\beta > -2$ and ε small enough. On the other hand, if $\beta > N-4$*

and ε is small, the reduced functional has no critical point; and if $\beta = N-4$, then the integral

$$\int_{-\infty}^{\infty} e^{-\frac{2(\beta+2)}{N-2}\rho} e^{-2\rho} \left(1 + e^{-\frac{4+2\alpha}{N-2}\rho}\right)^{-\frac{2(N-2)}{2+\alpha}} \, d\rho$$

does not converge, and since this is crucial in our reasoning, we cannot deal with both cases here.

Remark 1.4.4. *The linear non-degeneracy of equation (1.2.11) was established in [22] for the case $\alpha \geq 0$. This non-degeneracy result can be extended directly to the case $\alpha > -2$ following the ideas in [2], as the computations there are stable as $s \to 1$. Note that although the regularity of the solutions when $-2 < \alpha < 0$ differs from the case $\alpha \geq 0$ (see [18]), as far as we are concerned, this does not hinder our reasoning. Therefore, we may consider $\alpha > -2$.*

Remark 1.4.5. *Case $N = 3$ is significantly different in consideration of the generality established for $N \geq 4$. Thus, whether or not outcomes similar to those presented here are valid remains an open question for $N = 3$.*

Furthermore, we extend our study onto exterior domains, with the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p_\alpha^* - \varepsilon} - \lambda_\varepsilon |x|^{\beta} u & \text{in } \mathbb{R}^N \setminus B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \\ \limsup\limits_{|x|\to\infty} |x|^{N-2} u(x) < \infty, \end{cases} \tag{1.4.5}$$

where B_1 is the open unit ball centered at the origin. Pointing in this direction, our next goal is to study the existence of positive solutions that also take the form of a tower of bubbles of order α, but in this case it progressively flattens as ε tends to zero. The result we found is

Theorem 1.4.6. *Under the hypothesis of Theorem 1.4.2, there exist numbers $\eta_k > 0$ and $\bar{\varepsilon}_k > 0$ such that if $\eta \in (\eta_k, +\infty)$, $\varepsilon \in (0, \bar{\varepsilon}_k)$ and $\lambda_\varepsilon = \eta \varepsilon^{\frac{N+\beta}{N-2}}$, then there are constants $\mu_{i,\varepsilon,\alpha}$, for $i = 1, 2, \ldots, k$, which depend on k, N, η and ε, and a solution $u_{k,\varepsilon} : \mathbb{R}^N \setminus B_1 \to \mathbb{R}$ to problem (3.1.6) such that*

$$u_{k,\varepsilon}(x) = \sum_{i=1}^{k} P_{\mathbb{R}^N \setminus B_1} U_{\mu_{i,\varepsilon,\alpha}}(x) + |x|^{2-N} \varphi_\varepsilon(x), \quad x \in \mathbb{R}^N \setminus B_1, \tag{1.4.6}$$

where $\mu_{i,\varepsilon,\alpha} = M_i \varepsilon^{\frac{1-2i}{N-2}}$ for each $i = 1, 2, \ldots, k$, with $M_1, M_2, \ldots, M_k$, being positive constants that depend only on N, k and α, and $|x|^{2-N}\varphi_\varepsilon$ is a function of lower order than every $P_{\mathbb{R}^N \setminus B_1} U_{\mu_{j\varepsilon},\alpha}$ in $\mathbb{R}^N \setminus B_1$, with $\varphi_\varepsilon \to 0$ uniformly on compacts contained in $\mathbb{R}^N \setminus B_1$ as $\varepsilon \to 0$, with $P_{\mathbb{R}^N \setminus B_1} U_{\mu_{j\varepsilon},\alpha}$ being the projection onto $H_0^1(\mathbb{R}^N \setminus B_1)$ of the function $U_{\mu_{j\varepsilon},\alpha}$, for $j = 1, 2, \ldots, k$.

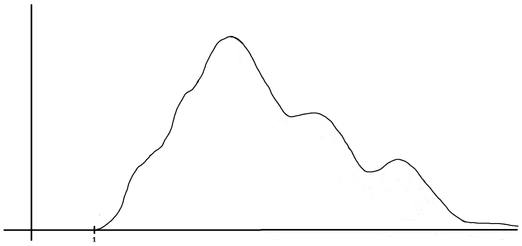

Fig. 3. Function (1.4.6) taking the form of a tower of bubbles, choosing $k = 3$, $\alpha = 10$, $N = 6$ and $\varepsilon = 0.5$.

Remark 1.4.7. *Working on the exterior of the unit ball allows us to use a Taylor expansion of a certain functional in a bubble, thereby avoiding the difficulties that arise when trying to incorporate the regular part of Green's function in an exterior domain.*

1.5 Structure and method used

The proof of Theorem 1.4.1, 1.4.2, 1.4.6 are based on the well-known Lyapunov-Schmidt reduction method under the approach introduced by Del Pino, Felmer, and Musso [15]. First, to establish an initial approximation of the solution, an Emden-Fowler type change of variables is used. Next, the linearized problem is studied, which forms the basis for performing the finite-dimensional reduction process. Then, by employing a non-degeneracy result, the use of norms in weighted spaces, various integral estimates, and Banach's fixed-point theorem, the reduction process is completed, leading to the problem of finding critical points of a functional defined on $\mathbb{R}^k$, concluding with the aid of Topological Degree Theory.

Chapter 2

Positive solutions for some almost critical Brezis-Niremberg problem of fourth order in a bounded domain

2.1 Ansatz

In this section, we aim to identify a suitable ansatz for a function u that solves the following problem

$$\begin{cases} \Delta^2 u = u^{p+\varepsilon} + \lambda_\varepsilon u & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ \Delta u = u = 0 & \text{on } \partial B_1, \end{cases} \tag{2.1.1}$$

where B_1 is the open unit ball in $\mathbb{R}^N$, $N > 8$, $p = \frac{N+4}{N-4}$, $\varepsilon > 0$ is small and λ_ε is a positive constant depending on ε.

Consider the equation

$$\begin{cases} \Delta^2 u(x) = & u^p(x) & \text{in } \mathbb{R}^N, \\ u > & 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{2.1.2}$$

for which all positive radial and symmetric solutions are given by the *bubbles*

$$U_{\mu,2,0}(x) = \overline{\gamma}_{N,2} \left(\frac{\mu}{\mu^2 + |x|^2} \right)^{\frac{N-4}{2}}, \tag{2.1.3}$$

form some $\mu > 0$ and $\overline{\gamma}_{N,2} = [N(N-4)(N-2)(N+2)]^{\frac{N-4}{8}}$.
For convenience, we define

$$U(x) = U_{1,2,0}(x) = \overline{\gamma}_{N,2}(1 + |x|^2)^{-\frac{N-4}{2}}$$

and

$$\begin{aligned} U_\mu(x) = U_{\mu,2,0}(x) &= \overline{\gamma}_{N,2}\left(\tfrac{\mu}{\mu^2+|x|^2}\right)^{\frac{N-4}{2}} \\ &= \overline{\gamma}_{N,2}\left(\tfrac{\mu}{1+\mu^2|x|^2}\right)^{\frac{N-4}{2}} \\ &= \overline{\gamma}_{N,2}\mu^{\frac{N-4}{2}}(1+|\mu x|^2)^{-\frac{N-4}{2}} \\ &= \mu^{\frac{N-4}{2}}U(\mu x). \end{aligned}$$

An initial observation about the *bubbles* is that they do not satisfy the boundary condition of the problem (2.1.1). To establish an appropriate ansatz, it is natural to make a correction to the bubble U_μ by projecting it onto the Hilbert space $H_0^1(B_1)\cap H^2(B_1)$. Let $P_{B_1}U_\mu$ be the projection onto this space. Then $P_{B_1}U_\mu$ satisfies

$$\begin{cases} \Delta^2 P_{B_1}U_\mu = U_\mu^p & \text{in } B_1, \\ P_{B_1}U_\mu = 0 & \text{on } \partial B_1, \\ \Delta P_{B_1}U_\mu = 0 & \text{on } \partial B_1. \end{cases} \tag{2.1.4}$$

Consequently, we can define $\pi(x) := P_{B_1}U_\mu(x) - U_\mu(x)$, which is the unique solution of the problem

$$\begin{cases} \Delta^2\pi = 0 & \text{in } B_1, \\ \pi = -U_\mu & \text{on } \partial B_1, \\ \Delta\pi = -\Delta U_\mu & \text{on } \partial B_1 \end{cases} \tag{2.1.5}$$

and is given by

$$\pi(x) = \frac{\Delta U_\mu(1)}{2N} - U_\mu(1) - \frac{\Delta U_\mu(1)}{2N}|x|^2. \tag{2.1.6}$$

Proof. Let $\pi(x) = a + b|x|^2$. Using the Laplacian formula in polar coordinates, we have

$$\begin{aligned} \Delta\pi(|x|) &= \pi''(|x|) + \tfrac{(N-1)}{r}\pi'(|x|) \\ &= 2b + \tfrac{N-1}{|x|}\cdot 2b|x| \\ &= 2Nb, \end{aligned}$$

which implies $\Delta^2\pi(x) = \Delta(\Delta\pi(x)) = 0$. To determine a and b, we use the boundary conditions of the equation and solve the associated system. In other words, when $|x| = 1$, we have $a + b = -U_\mu(1)$ and $2Nb = -\Delta U_\mu(1)$, resulting in

$$a = \frac{\Delta U_\mu(1)}{2N} - U_\mu(1), \quad b = -\frac{\Delta U_\mu(1)}{2N}.$$

□

To continue in our search for an ansatz, we will transform problem (2.1.1) into an equivalent one, using an adaptation of the Emden-Fowler transformation. For this, some preliminary calculations are needed.

2.1.1 Preliminaries

Given that $u(x) = u(|x|) = u(r)$, the Laplacian in polar coordinates is expressed as

$$\Delta u(r) = u''(r) + \frac{(N-1)}{r}u'(r),$$

which implies

$$\begin{aligned}
\Delta^2 u = &\ \Delta(\Delta u)\\
= &\ \Delta\left(u''(r) + \tfrac{N-1}{r}u'(r)\right)\\
= &\ \left(u'' + \tfrac{N-1}{r}u'\right)'' + \tfrac{N-1}{r}\left(u'' + \tfrac{N-1}{r}u'\right)'\\
= &\ u^{(4)} - \left(\tfrac{N-1}{r^2}u'\right)' + \left(\tfrac{N-1}{r}u''\right)' + \tfrac{N-1}{r}\left(u''' - \tfrac{N-1}{r^2}u' + \tfrac{N-1}{r}u''\right)\\
= &\ u^{(4)} + \tfrac{2(N-1)}{r^3}u' - \tfrac{N-1}{r^2}u'' - \tfrac{N-1}{r^2}u'' + \tfrac{N-1}{r}u''' + \tfrac{N-1}{r}u''' - \tfrac{(N-1)^2}{r^3}u' + \tfrac{(N-1)^2}{r^2}u''\\
= &\ u^{(4)} + \tfrac{2(N-1)}{r}u''' + \tfrac{(N-1)(N-3)}{r^2}u'' - \tfrac{(N-1)(N-3)}{r^3}u'.
\end{aligned}$$

Thus, (2.1.1) becomes

$$u^{(4)} + \frac{2(N-1)}{r}u''' + \frac{(N-1)(N-3)}{r^2}u'' - \frac{(N-1)(N-3)}{r^3}u' = u^{p+\varepsilon} + \lambda_\varepsilon u. \tag{2.1.7}$$

Consider now the transformation

$$u(r) = Ce^{at}v(t), \quad \text{with } r = e^t$$

and a a constant to be determined later. Direct calculations show that

$$\begin{aligned}
u'(r) &= \left(Cae^{at}v(t) + Ce^{at}v'(t)\right)\frac{1}{e^t}\\
&= Ce^{t(a-1)}\left(av(t) + v'(t)\right),
\end{aligned}$$

$$\begin{aligned}
u''(r) &= C(a-1)e^{t(a-1)}\frac{1}{e^t}\left(av(t) + v'(t)\right) + Ce^{t(a-1)}\left(av'(t) + v''(t)\right)\frac{1}{e^t}\\
&= Ce^{t(a-2)}\left[(a^2-a)v(t) + (2a-1)v'(t) + v''(t)\right],
\end{aligned}$$

$$\begin{aligned}
u'''(r) = &\ C(a-2)e^{t(a-2)}\frac{1}{e^t}\left[(a^2-a)v(t) + (2a-1)v'(t) + v''(t)\right]\\
&+ Ce^{t(a-2)}\left[(a^2-a)v'(t) + (2a-1)v''(t) + v'''(t)\right]\frac{1}{e^t}\\
= &\ Ce^{t(a-3)}\left[(a^3 - 3a^2 + 2a)v(t) + (3a^2 - 6a + 2)v'(t) + (3a-3)v''(t) + v'''(t)\right],
\end{aligned}$$

$$
\begin{aligned}
u^{(4)} &= C(a-3)e^{t(a-3)}\frac{1}{e^t}\left[(a^3-3a^2+2a)v(t)+(3a^2-6a+2)v'(t)+(3a-3)v''(t)+v'''(t)\right] \\
&\quad + Ce^{t(a-3)}\left[(a^3-3a^2+2a)v'(t)+(3a^2-6a+2)v''(t)+(3a-3)v'''(t)+v^{(4)}(t)\right]\frac{1}{e^t} \\
&= Ce^{t(a-4)}\left[(a^4-6a^3+11a^2-6a)v(t)+(4a^3-18a^2+22a-6)v'(t)\right. \\
&\quad \left.+(6a^2-18a+11)v''(t)+(4a-6)v'''(t)+v^{(4)}(t)\right].
\end{aligned}
$$

Replacing on (2.1.7)

$$
Ce^{t(a-4)}(v^{(4)}(t)+C_3v'''(t)+C_2v''(t)+C_1v'(t)+C_0v(t)) = C^{p+\varepsilon}e^{at(p+\varepsilon)}v^{p+\varepsilon}(t)+\lambda_\varepsilon Ce^{at}v(t), \quad (2.1.8)
$$

where

$$
\begin{aligned}
C_3 &= (4a-6)+2(N-1), \\
C_2 &= (6a^2-18a+11)+2(N-1)(3a-3)+(N-1)(N-3), \\
C_1 &= (4a^3-18a^2+22a-6)+2(N-1)(3a^2-6a+2)+(N-1)(N-3)(2a-1) \\
&\quad -(N-1)(N-3), \\
C_0 &= (a^4-6a^3+11a^2-6a)+2(N-1)(a^3-3a^2+2a)+(N-1)(N-3)(a^2-a) \\
&\quad -(N-1)(N-3)a.
\end{aligned}
$$

Taking $\varepsilon = 0$, we seek the equality $a-4 = ap$, hence $a = -\frac{N-4}{2}$. Substituting into the constants above, we find

$$C_3 = 0,$$

$$C_2 = -\frac{N^2}{2}+2N-4,$$

$$C_1 = 0,$$

$$C_0 = \frac{N^2}{16}(N-4)^2,$$

and therefore, (2.1.8) transforms into

$$
v^{(4)}(t)-\frac{1}{2}(N^2-4N+8)v''(t)+\frac{N^2}{16}(N-4)^2v(t) = C^{p-1+\varepsilon}e^{at\varepsilon}v^{p+\varepsilon}+\lambda_\varepsilon e^{4t}v(t). \quad (2.1.9)
$$

Letting $v_*(\rho) = v(t)$, with $\rho = at$, we have

$$
a^4v_*^{(4)}(\rho)-\frac{a^2}{2}(N^2-4N+8)v_*''(\rho)+\frac{N^2}{16}(N-4)^2v_*(\rho) = C^{p-1+\varepsilon}e^{\varepsilon\rho}v_*^{p+\varepsilon}(\rho)+\lambda_\varepsilon e^{\frac{4\rho}{a}}v_*(\rho).
$$

Choosing C such that $C^{p-1+\varepsilon} = a^4$, we find

$$\begin{aligned} C =&\ a^{\frac{4}{p-1+\varepsilon}} \\ =&\ \left(\frac{N-4}{2}\right)^{\frac{4}{p-1+\varepsilon}}, \end{aligned}$$

yielding to

$$v_*^{(4)}(\rho) - C_4 v_*''(\rho) + C_5 w(\rho) = e^{\varepsilon\rho} v_*^{p+\varepsilon}(\rho) + \lambda_\varepsilon \left(\frac{2}{N-4}\right)^4 e^{\frac{4\rho}{a}} v_*(\rho),$$

with

$$C_4 = \frac{1}{2a^2}(N^2 - 4N + 8) = \frac{2(N^2 - 4N + 8)}{(N-4)^2},$$

$$C_5 = \frac{N^2}{16a^4}(N-4)^2 = \frac{N^2}{(N-4)^2}.$$

Notice that $1 - C_4 + C_5 = 0$. Additionally, note that

$$u(r) = \left(\frac{N-4}{2}\right)^{\frac{4}{p-1+\varepsilon}} e^{\rho} v_*(\rho),$$

thus, the Emden-Fowler transformation sought is

$$v_*(\rho) := \mathcal{T}(u)(\rho) = \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho} u(e^{-\frac{2}{N-4}\rho}), \quad \text{with } r = e^{-\frac{2}{N-4}\rho}. \tag{2.1.10}$$

For the boundary conditions, we know that $u(x) = 0$ on ∂B_1 means $v_*(0) = 0$ and from

$$\Delta u(1) = u''(1) + (N-1)u'(1) = 0,$$

we have

$$(N-4)v_*''(0) - 4v_*'(0) - N v_*(0) = 0.$$

Therefore, problem (2.1.1) becomes the equivalent problem

$$\begin{cases} L(v_*) = & e^{\varepsilon\rho} v_*^{p+\varepsilon}(\rho) + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} v_*(\rho) \quad \text{in } (0,\infty), \\ v_*(\rho) > & 0 \quad \text{in } (0,\infty), \\ v_*(0) = & 0, \\ (N-4)v_*''(0) - 4v_*'(0) = & 0, \end{cases} \tag{2.1.11}$$

where

$$a_\varepsilon = \left(\frac{2}{N-4}\right)^{\frac{4}{p-1+\varepsilon}},$$

$$L(v_*) = v_*^{(4)}(\rho) - C_4 v_*''(\rho) + C_5 v_*(\rho) \tag{2.1.12}$$

and the associate energy functional of (2.1.11) is

$$E_\varepsilon(v_*) = I_\varepsilon(v_*) - K_\varepsilon(v_*), \tag{2.1.13}$$

where

$$I_\varepsilon(v_*) = \frac{1}{2}\int_0^\infty (v_*''(\rho)^2 + C_4 v_*'(\rho)^2 + C_5 v_*(\rho)^2)d\rho + \frac{4}{N-4}v_*'(0)^2 - \frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon\rho} v_*^{p+\varepsilon+1}(\rho)d\rho, \tag{2.1.14}$$

and

$$K_\varepsilon(v_*) = \lambda_\varepsilon a_\varepsilon \frac{1}{2}\int_0^\infty e^{-\frac{8}{N-4}\rho} v_*(\rho)^2\, d\rho. \tag{2.1.15}$$

Proof. Let $w, v \in H_0^1(0,\infty) \cap H^2(0,\infty)$, with w satisfying (2.1.11). By direct calculations, we have

$$\begin{aligned}\langle DE_\varepsilon(w), v\rangle = & \int_0^\infty (w''(\rho)v''(\rho) + C_4 w'(\rho)v'(\rho) + C_5 w(\rho)v(\rho))\, d\rho \ + \frac{4}{N-4}w'(0)v'(0) \\ & - \int_0^\infty e^{\varepsilon\rho} w^{p+\varepsilon}(\rho)v(\rho)\, d\rho - \lambda_\varepsilon a_\varepsilon \int_0^\infty e^{-\frac{8}{N-4}\rho} w(\rho)v(\rho)\, d\rho.\end{aligned}$$

Integrating by parts, leads us to

$$\begin{aligned}\langle DE_\varepsilon(w), v\rangle &= -w''(0)v'(0) + w'''(0)v(0) - C_4 w'(0)v(0) + \frac{4}{N-4}w'(0)v'(0) \\ &\quad + \int_0^\infty \Big(w^{(4)}(\rho) - C_4 w''(\rho) + C_5 w(\rho)\Big)v(\rho)\, d\rho \ - \int_0^\infty e^{\varepsilon\rho} w^{p+\varepsilon}(\rho)v(\rho)\, d\rho \\ &\quad - \lambda_\varepsilon a_\varepsilon \int_0^\infty e^{-\frac{8}{N-4}\rho} w(\rho)v(\rho)\, d\rho \\ &= -w''(0)v'(0) + \frac{4}{N-4}w'(0)v'(0) \\ &\quad + \int_0^\infty \Big(w^{(4)}(\rho) - C_4 w''(\rho) + C_5 w(\rho)\Big)v(\rho)\, d\rho \\ &\quad - \int_0^\infty e^{\varepsilon\rho} w^{p+\varepsilon}(\rho)v(\rho)\, d\rho - \lambda_\varepsilon a_\varepsilon \int_0^\infty e^{-\frac{8}{N-4}\rho} w(\rho)v(\rho)\, d\rho \\ &= -w''(0)v'(0) + \frac{4}{N-4}w'(0)v'(0).\end{aligned}$$

Choosing $v \in H_0^1(0,\infty) \cap H^2(0,\infty)$ such that $v'(0) \neq 0$, then $\langle DE_\varepsilon(w), v\rangle = 0$ and it follows that w is a critical point of I_ε.

In the other hand, if $\langle DE_\varepsilon(w), v\rangle = 0$, let $v \in C_0^\infty$. By integrating by parts, we have

$$\langle DE_\varepsilon(w), v\rangle = \int_0^\infty \Big(w^{(4)}(\rho) - C_4 w''(\rho) + C_5 w(\rho) - e^{\varepsilon\rho} w^{p+\varepsilon} - \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} w^{p+\varepsilon}(\rho)\Big)v(\rho)\, d\rho,$$

thus $w^{(4)}(\rho) - C_4 w''(\rho) + C_5 w(\rho) - e^{\varepsilon\rho} w^{p+\varepsilon} - \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} w^{p+\varepsilon}(\rho) = 0$. Moreover, if $v \in H_0^1(0,\infty) \cap$

$H^2(0,\infty)$, then once again we can integrate by parts to obtain

$$\langle DE_\varepsilon(w), v\rangle = -w''(0)v'(0) + \frac{4}{N-4}w'(0)v'(0).$$

Choosing $v'(0) \neq 0$, we conclude that $(N-4)w''(0) - 4w'(0) = 0$ and this in turn means that w satisfies (2.1.11). Henceforth, w is a critical point of E_ε if and only if, w satisfies (2.1.11). $\square$

Now that (2.1.1) has been transformed into the equivalent problem (2.1.11), it is of interest to obtain information from the transformed problem that allows us to find a suitable ansatz for (2.1.11). In this case, we consider the limit problem (2.1.2) under the transformation $\mathcal{T}$ defined in (2.1.10) and the solutions of the transformed limit problem as the main element of the ansatz. Specifically, we are interested in studying the function

$$W(\rho) := \mathcal{T}\left[U\right](\rho) = \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho}U(e^{-\frac{2}{N-4}\rho}), \tag{2.1.16}$$

which, after carrying out the transformation, takes the form

$$\begin{aligned} W(\rho) = & \left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho}\overline{\gamma}_{N,2}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\ = & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}}, \end{aligned}$$

with $K_N = \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}}\left[N(N-4)(N-2)(N+2)\right]^{\frac{N-4}{8}}$.
Also, note that if $\mu \neq 1$, then

$$\begin{aligned} W_\xi(\rho) := \mathcal{T}\left[U_\mu\right](\rho) = & K_N e^{-\rho}\mu^{\frac{N-4}{2}}\left(1+\mu^2 e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\ = & K_N e^{-(\rho-\frac{N-4}{2}\ln(\mu))}\left(1+e^{-\frac{4}{N-4}(\rho-\frac{N-4}{2}\ln(\mu))}\right)^{-\frac{N-4}{2}} \\ = & W(\rho-\tfrac{N-4}{2}\ln(\mu)). \end{aligned}$$

Moreover, the function $W(\rho)$ satisfies the equation

$$W^{(4)}(\rho) - C_4 W''(\rho) + C_5 W(\rho) - W^p = 0. \tag{2.1.17}$$

Proof. Indeed, by straightforward calculations

$$\begin{aligned} W'(\rho) = & -K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} + K_N e^{-\rho}\left(-\tfrac{N-4}{2}\right)\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\ = & -K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} + 2K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}, \end{aligned}$$

$$
\begin{aligned}
W''(\rho) =\ & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} - K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(-\tfrac{N-4}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& + 2K_N e^{-\frac{N}{N-4}\rho}\left(-\tfrac{N}{N-4}\right)\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \\
& + 2K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\left(-\tfrac{N-2}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
=\ & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} - 4K_N\left(\tfrac{N-2}{N-4}\right)e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \\
& + 4K_N\left(\tfrac{N-2}{N-4}\right)e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}},
\end{aligned}
$$

$$
\begin{aligned}
W'''(\rho) =\ & -K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} + K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(-\tfrac{N-4}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& + 4K_N\left(\tfrac{N}{N-4}\right)\left(\tfrac{N-2}{N-4}\right)e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \\
& - 4K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\left(\tfrac{N-2}{N-4}\right)\left(-\tfrac{N-2}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& + 4K_N\left(\tfrac{N-2}{N-4}\right)\left(-\tfrac{N+4}{N-4}\right)e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}} \\
& + 4K_N\left(\tfrac{N-2}{N-4}\right)e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}}\left(-\tfrac{N}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
=\ & -K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} + 2K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(1+\tfrac{2N(N-2)}{(N-4)^2}\right) \\
& - 4K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\left(\tfrac{3N(N-2)}{(N-4)^2}\right) + 8K_N\left(\tfrac{N(N-2)}{(N-4)^2}\right)e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
W^{(4)}(\rho) =\ & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} - K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(-\tfrac{N-4}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& - 2K_N\left(\tfrac{N}{N-4}\right)e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}}\left(1+\tfrac{2N(N-2)}{(N-4)^2}\right) \\
& + 2K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\left(1+\tfrac{2N(N-2)}{(N-4)^2}\right)\left(-\tfrac{N-2}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& + 4K_N\left(\tfrac{N+4}{N-4}\right)e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\left(\tfrac{3N(N-2)}{(N-4)^2}\right) \\
& - 4K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}}\left(\tfrac{3N(N-2)}{(N-4)^2}\right)\left(-\tfrac{N}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho} \\
& - 8K_N\left(\tfrac{N(N-2)}{(N-4)^2}\right)\left(\tfrac{N+8}{N-4}\right)e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}} \\
& + 8K_N\left(\tfrac{N(N-2)}{(N-4)^2}\right)e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+4}{2}}\left(-\tfrac{N+2}{2}\right)\left(-\tfrac{4}{N-4}\right)e^{-\frac{4}{N-4}\rho}
\end{aligned}
$$

$$
\begin{aligned}
= \; & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} - 8K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \frac{(N-2)(N^2-4N+8)}{(N-4)^3} \\
& + 8K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}} \frac{(N-2)(3N^2+8)}{(N-4)^3} - 32K_N e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}} \frac{N(N-2)(N+2)}{(N-4)^3} \\
& + 16K_N e^{-\frac{N+12}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+4}{2}} \frac{N(N-2)(N+2)}{(N-4)^3}.
\end{aligned}
$$

Replacing on (2.1.17) we get,

$$
\begin{aligned}
W^{(4)} - C_4 W'' + C_5 W = \; & K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\
& - 8K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \frac{(N-2)(N^2-4N+8)}{(N-4)^3} \\
& + 8K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}} \frac{(N-2)(3N^2+8)}{(N-4)^3} \\
& - 32K_N e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}} \frac{N(N-2)(N+2)}{(N-4)^3} \\
& + 16K_N e^{-\frac{N+12}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+4}{2}} \frac{N(N-2)(N+2)}{(N-4)^4} \\
& - \frac{2(N^2-4N+8)}{(N-4)^2}\Bigg[K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\
& - 4K_N e^{-\frac{N}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} \frac{N-2}{N+2} \\
& + 4K_N \frac{N-2}{N-4} e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}}\Bigg] \\
& + \frac{N^2}{(N-4)^2} K_N e^{-\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\
= \; & 8K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N}{2}} \frac{(N-2)}{(N-4)^3}\left(2N^2+4N\right) \\
& - 32K_N e^{-\frac{N+8}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+2}{2}} \frac{N(N-2)(N+2)}{(N-4)^3} \\
& + 16K_N e^{-\frac{N+12}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+4}{2}} \frac{N(N-2)(N+2)}{(N-4)^3} \\
= \; & 16K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N+4}{2}} \frac{N(N-2)(N+2)}{(N-4)^3}\Bigg[\left(1+e^{-\frac{4}{N-4}\rho}\right)^2 \\
& - 2e^{-\frac{4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right) + e^{-\frac{8}{N-4}\rho}\Bigg] \\
= \; & K_N e^{-\frac{N+4}{N-4}\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-(\frac{N-4}{2})(\frac{N+4}{N-4})} 16\frac{N(N-2)(N+2)}{(N-4)^3}\Bigg[\left(1+e^{-\frac{4}{N-4}\rho}\right) \\
& - e^{-\frac{4}{N-4}\rho}\Bigg]^2 \\
= \; & K_N e^{-p\rho}\left(1+e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}p} K_N^{p-1} \\
= \; & W^p.
\end{aligned}
$$

□

Just as the function U_μ did not satisfy the boundary conditions of the problem (2.1.1), the function $W(\rho)$ does not satisfy them either, so we are also interested in the function $\Pi_\xi(\rho)$, defined as $\Pi_\xi(\rho) := \mathcal{T}(\pi_\mu)(\rho)$, with $\xi = \frac{N-4}{2}\log\mu$; and it verifies the problem

$$\begin{cases} L(\Pi_\xi) = & 0 \quad \text{in } (0,\infty), \\ \Pi_\xi(0) = & -W_\xi(0), \\ \Delta\Pi_\xi(0) = & -\Delta W_\xi(0) \end{cases} \tag{2.1.18}$$

and is given by

$$\Pi_\xi(\rho) = A_1 e^{-\rho} + A_2 e^{-\frac{N-8}{N-4}\rho}, \tag{2.1.19}$$

where

$$A_1 = \frac{(N-4)^2}{8N} W''_\xi(0) - \frac{N-4}{2N} W'_\xi(0) - \frac{N-4}{8} W_\xi(0)$$

and

$$A_2 = -\frac{(N-4)^2}{8N} W''_\xi(0) + \frac{N-4}{2N} W'_\xi(0) + \frac{N+4}{8} W_\xi(0).$$

Proof. From (2.1.6), let $\pi_\mu = a + b|x|^2$, where

$$a = \frac{\Delta U_\mu(1)}{2N} - U_\mu(1) \quad \text{and} \quad b = -\frac{\Delta U_\mu(1)}{2N}.$$

By (2.1.10), we have

$$\begin{aligned} \Pi_\xi = \mathcal{T}(\pi_\mu)(\rho) &= \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho} \pi_\mu(e^{-\frac{2}{N-4}\rho}) \\ &= \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho} \left(a + b e^{-\frac{4}{N-4}\rho}\right) \\ &= \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho} a + \left(\frac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\frac{N-8}{N-4}\rho} b. \end{aligned}$$

Notice that $U_\mu(1)$ is equivalent to $\left(\frac{N-4}{2}\right)^{\frac{N-4}{2}} W_\xi(0)$. Moreover, given $U_\mu(x) = U_\mu(|x|) = U_\mu(r)$, the Laplacian in polar coordinates is expressed as

$$\Delta U_\mu(r) = U''_\mu(r) + \frac{N-1}{r} U'_\mu(r).$$

From (2.1.16), we can use direct calculations to obtain

$$\begin{aligned} U'_\mu(r) =& \left(\tfrac{N-4}{2}\right)^{\frac{N-4}{2}} e^{\rho} \left(-\tfrac{N-4}{2}\right) e^{-\frac{2}{N-4}} W_\xi(\rho) + \left(\tfrac{N-4}{2}\right)^{\frac{N-4}{2}} e^{\rho} W'_\xi(\rho) \left(-\tfrac{N-4}{2}\right) e^{-\frac{2}{N-4}} \\ =& -\left(\tfrac{N-4}{2}\right)^{\frac{N-2}{2}} e^{\frac{N-2}{N-4}\rho} (W'_\xi(\rho) + W_\xi(\rho)) \end{aligned}$$

and

$$\begin{aligned} U_\mu''(r) =& -\left(\tfrac{N-4}{2}\right)^{\frac{N-2}{2}} e^{\frac{N-2}{N-4}\rho}\tfrac{N-2}{N-4}\left(-\tfrac{N-4}{2}\right)e^{-\frac{2}{N-4}}(W_\xi'(\rho)+W_\xi(\rho)) \\ & -\left(\tfrac{N-4}{2}\right)^{\frac{N-2}{2}} e^{\frac{N-2}{N-4}\rho}(W_\xi''(\rho)+W_\xi'(\rho))\left(-\tfrac{N-4}{2}\right)e^{-\frac{2}{N-4}} \\ =& \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}} e^{\rho}\left(W_\xi''(\rho)+\tfrac{2(N-3)}{N-4}W_\xi'(\rho)+\tfrac{N-2}{N-4}W_\xi(\rho)\right), \end{aligned}$$

where $r = e^{-\frac{2}{N-4}\rho}$. Thus, upon substitution, we obtain

$$\begin{aligned} \Delta U_\mu(1) =& \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}}\left(W_\xi''(0)+\tfrac{2(N-3)}{N-4}W_\xi'(0)+\tfrac{2(N-3)}{N-4}W_\xi(0)\right) \\ & -(N-1)\left(\tfrac{N-4}{2}\right)^{\frac{N-2}{2}}(W_\xi'(0)+W_\xi(0)) \\ =& \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}}\left(W_\xi''(0)-\tfrac{4}{N-4}W_\xi'(0)-\tfrac{N}{N-4}W_\xi(0)\right). \end{aligned}$$

Therefore, we can get

$$\begin{aligned} A_2 =& \left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}} b \\ =& -\left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}}\tfrac{\Delta U_\mu(1)}{2N} \\ =& -\tfrac{(N-4)^2}{8N}W_\xi''(0)+\tfrac{N-4}{2N}W_\xi'(0)+\tfrac{N-4}{8}W_\xi(0). \end{aligned}$$

For A_1, notice that

$$\begin{aligned} A_1 =& \left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}} a \\ =& \left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}}\left(\tfrac{\Delta U_\mu(1)}{2N}-U_\mu(1)\right) \\ =& \tfrac{(N-4)^2}{8N}W_\xi''(0)-\tfrac{N-4}{2N}W_\xi'(0)-\tfrac{N-4}{8}W_\xi(0)-\left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}}\left(\tfrac{N-4}{2}\right)^{\frac{N-4}{2}}W_\xi(0) \\ =& \tfrac{(N-4)^2}{8N}W_\xi''(0)-\tfrac{N-4}{2N}W_\xi'(0)-\tfrac{N+4}{8}W_\xi(0). \end{aligned}$$

□

To finish this section, we also consider constants $0 < \xi_1 < \cdots < \xi_k$ in $\mathbb{R}$ and constants μ_i, with $i = 1, \ldots, k$ such that

$$\xi_i = \frac{N-4}{2}\log\mu_i. \tag{2.1.20}$$

Therefore, after all the study conducted in this section, it is reasonable to seek solutions of (2.1.11) that have the form

$$v_{k,\varepsilon}(\rho) = \sum_{i=1}^{k}\left(W_{\xi_i}(\rho)+\Pi_{\xi_i}(\rho)\right)+\phi(\rho), \tag{2.1.21}$$

where ϕ is a function expected to have small order in some sense to be specified, and depends on the positive values ξ_i.

2.2 Expansion of the reduced energy

In this section, we will consider v_* given by (2.1.21). To simplify notation, for $i = 1, \ldots, k$ and positive values $\xi_i \in \mathbb{R}$, from now on we will use the following notation

$$W_i := W_{\xi_i} \quad \Pi_i := \Pi_{\xi_i} \quad V_i := W_i + \Pi_i \quad \text{and } V = \sum_{i=1}^{k} V_i, \tag{2.2.1}$$

so that (2.1.21) can be written as $V + \phi$. Furthermore, we choose the parameters ξ_i using the μ_i conveniently, namely

$$\xi_{\varepsilon,1} = -\frac{1}{2}\log\varepsilon + \log\mu_1, \quad \xi_{\varepsilon,i+1} - \xi_{\varepsilon,i} = -\log\varepsilon - \log\mu_{i+1}, \quad \text{for } i = 1, 2, \ldots, k-1. \tag{2.2.2}$$

Before stating the main lemma of this chapter, we need an appropriate estimate for the previously defined functions.

Proposition 2.2.1. *The functions defined in (2.2.1) satisfy the following estimates*

$$|W_i(\rho)| \le Ce^{-|\rho-\xi_i|}, \quad |\Pi_i(\rho)| \le Ce^{-|\rho-\xi_i|} \quad \text{and} \quad |V_i(\rho)| \le Ce^{-|\rho-\xi_i|},$$

where C is a positive constant depending only on N.

Proof. Given that the function W is even, we have

$$\begin{aligned} W_i(\rho) = &\ W_i(-\rho) \\ = &\ W(|\rho-\xi_i|) \\ = &\ K_N e^{-|\rho-\xi_i|}\left(1 + e^{-\frac{4}{N-4}|\rho-\xi_i|}\right)^{-\frac{N-4}{2}} \\ \le &\ K_N e^{-|\rho-\xi_i|}, \end{aligned}$$

since $\left(1 + e^{-\frac{4}{N-4}|\rho-\xi_i|}\right)^{-\frac{N-4}{2}} \le 1$.

For $\Pi_i(\rho)$, from (2.1.19) we know that this function can be written as a combination of $W_i(0)$, $W_i'(0)$ and $W_i''(0)$. Notice that, from previous calculation, we have

$$\begin{aligned} |W_i'(0)| = &\ |W'(-\xi_i)| \\ \le &\ K_N e^{-\xi_i}\left(1 + e^{-\frac{4}{N-4}\xi_i}\right)^{-\frac{N-4}{2}} + 2K_N e^{-\frac{N}{N-4}\xi_i}\left(1 + e^{-\frac{4}{N-4}\xi_i}\right)^{-\frac{N-2}{2}} \\ \le &\ Ce^{-\xi_i} \end{aligned}$$

In the same way, we can estimate

$$W''(0) \le Ce^{-\xi_i}$$

. Therefore

$$\begin{aligned} |\Pi_i(\rho)| \le &\ |A_1|e^{-\rho} + |A_2|e^{-\frac{N-8}{N-4}\rho} \\ \le &\ Ce^{-\xi_i}e^{-\rho} + Ce^{-\xi_i}e^{-\frac{N-8}{N-4}\rho} \\ \le &\ Ce^{-(\rho+\xi_i)} \\ \le &\ Ce^{-|\rho-\xi_i|}. \end{aligned}$$

Finally, for V, it suffices to use its definition to conclude that

$$\begin{aligned} |V_i(\rho)| \leq &\ |W_i(\rho)| + |\Pi_i(\rho)| \\ \leq &\ Ce^{-|\rho-\xi_i|}. \end{aligned}$$

□

The objective of this section, is to prove the following Lemma.

Lemma 2.2.2. *Let $k \in \mathbb{N}$, $k \geq 2$. Let $\delta > 0$ and assume that*

$$\delta < \mu_i < \delta^{-1} \quad \textit{for } i = 1, 2, \ldots, k. \tag{2.2.3}$$

Suppose that $\lambda_\varepsilon = \eta\varepsilon^{\frac{N-8}{N-4}}$, for some $\eta > 0$ sufficiently large. Then, for the V defined in (2.2.1) and points ξ_i in (2.2.2), there exist positive numbers a_1, a_2, a_3, a_4, a_5, and a_6 depending only on N such that

$$E_\varepsilon(V) = ka_1 + \varepsilon\Psi_k(\boldsymbol{\mu}) + \frac{k^2}{2}(\varepsilon\log\varepsilon)a_2 + k\varepsilon a_4 + \varepsilon\Theta_\varepsilon(\boldsymbol{\mu}),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$,

$$\Psi_k(\boldsymbol{\mu}) = a_2\sum_{i=2}^{k}(k-i+1)\log\mu_i - ka_2\log\mu_1 - a_3\sum_{i=2}^{k}\mu_i + a_5\frac{1}{\mu_1^2} - a_6\eta\mu_1^{-\frac{8}{N-4}} \tag{2.2.4}$$

and $\Theta_\varepsilon(\boldsymbol{\mu}) \to 0$ as $\varepsilon \to 0$ uniformly with respect to μ_i satisfying (2.2.3).

Proof. By Taylor, we can expand

$$I_\varepsilon(V) = I_0(V) + \varepsilon\left(\frac{d}{d\varepsilon}I_\varepsilon(V)\bigg|_{\varepsilon=0}\right) + o(\varepsilon), \tag{2.2.5}$$

where

$$\begin{aligned} \frac{d}{d\varepsilon}I_\varepsilon(V) = &\ \frac{1}{(p+\varepsilon+1)^2}\int_0^\infty e^{\varepsilon\rho}V(\rho)^{p+\varepsilon+1}\,d\rho - \frac{1}{p+\varepsilon+1}\int_0^\infty \rho e^{\varepsilon\rho}V^{p+\varepsilon+1}\,d\rho \\ &-\frac{1}{p+\varepsilon+1}\int_0^\infty e^{\varepsilon\rho}V^{p+\varepsilon+1}(\rho)\ln(v_*)\,d\rho \end{aligned}$$

and this in turn implies

$$\begin{aligned} \left(\frac{d}{d\varepsilon}I_\varepsilon(V)\bigg|_{\varepsilon=0}\right) = &\ \frac{1}{(p+1)^2}\int_0^\infty V^{p+1}(\rho)\,d\rho - \frac{1}{p+1}\int_0^\infty \rho V(\rho)^{p+1}\,d\rho \\ &-\frac{1}{p+1}\int_0^{p+1} V(\rho)^{p+1}\ln(V)\,d\rho. \end{aligned} \tag{2.2.6}$$

Define

$$\zeta_1 = 0, \quad \zeta_{i+1} = \frac{1}{2}(\xi_{i+1} + \xi_i), i = 1, \ldots, k-1 \quad \text{and} \;\; \zeta_k + 1 = \infty. \tag{2.2.7}$$

Note that

$$\begin{aligned}\varepsilon\int_{\zeta_i}^{\zeta_{i+1}} V^{p+1}(\rho)\,d\rho = {} & \varepsilon\int_{\mathbb{R}} W_i(\rho)^{p+1}\,d\rho + \varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V(\rho)^{p+1} - V_i(\rho)^{p+1}\right)\,d\rho \\ & + \varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V_i(\rho)^{p+1} - W_i(\rho)^{p+1}\right)\,d\rho - \varepsilon\int_{-\infty}^{\zeta_i} W_i(\rho)^{p+1}\,d\rho \\ & - \varepsilon\int_{\zeta_{i+1}}^{\infty} W_i(\rho)^{p+1}\,d\rho.\end{aligned} \tag{2.2.8}$$

Knowing that $W_i(\rho)$ is even and Proposition 2.2.1, then

$$\begin{aligned}\int_{\zeta_{i+1}}^{\infty} W_i(\rho)^{p+1}\,d\rho \leq {} & C\int_{\zeta_{i+1}}^{\infty} e^{-(p+1)|\rho-\xi_i|}\,d\rho \\ = {} & Ce^{-(p+1)(\zeta_{i+1}-\xi_i)} \\ = {} & Ce^{-\frac{1}{2}(p+1)(\xi_{i+1}-\xi_i)} \\ = {} & C\mu_{i+1}^{\frac{p+1}{2}}\varepsilon^{\frac{p+1}{2}}.\end{aligned}$$

In the same way, we can take $y = \rho - \xi_i$

$$\begin{aligned}\int_{-\infty}^{\zeta_i} W_i(\rho)^{p+1}\,d\rho = {} & \int_{-\infty}^{\zeta_i-\xi_i} W(y)^{p+1}\,dy \\ = {} & -\int_{\infty}^{\xi_i-\zeta_i} W(\rho)^{p+1}\,d\rho \\ \leq {} & C\int_{\xi_i-\zeta_i}^{\infty} e^{-(p+1)\rho}\,d\rho \\ = {} & Ce^{(p+1)(\zeta_i-\xi_i)}.\end{aligned}$$

If $i = 1$, then

$$\int_{-\infty}^{\zeta_1} W_i(\rho)^{p+1}\,d\rho \leq Ce^{(p+1)(\zeta_1-\xi_1)} = Ce^{-(p+1)\xi_1} = Ce^{(p+1)(\frac{1}{2}\log\varepsilon - \log\mu_1)} = C\varepsilon^{\frac{p+1}{2}}\mu_1^{-(p+1)}.$$

On the other hand, if $i \geq 2$,

$$\int_{-\infty}^{\zeta_i} W_i(\rho)^{p+1}\,d\rho \leq Ce^{(p+1)(\zeta_i-\xi_i)} = Ce^{-\frac{1}{2}(p+1)(\xi_i-\xi_{i-1})} = C\varepsilon^{\frac{p+1}{2}}\mu_i^{\frac{p+1}{2}}.$$

Now, define $f(t) = \left(V_i(\rho) + t\sum\limits_{j=1,j\neq i}^{k} V_j(\rho)\right)^{p+1}$. By the Mean value theorem

$$\varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V(\rho)^{p+1} - V_i(\rho)^{p+1}\right)d\rho = \varepsilon\int_{\zeta_i}^{\zeta_{i+1}}(p+1)\left(V_i(\rho) + \nu\sum_{j=1,j\neq i}^{k} V_j(\rho)\right)^{p}\sum_{j=1,j\neq i}^{k} V_j(\rho)\,d\rho,$$

for some $\nu \in (0,1)$. Moreover, as $\rho \in [\zeta_i, \zeta_{i+1}]$ and $V_j(\rho) \leq Ce^{-|\rho-\xi_j|}$, we have

$$\begin{aligned}
\sum_{j=1, j\neq i}^{k} V_j(\rho) \leq & \sum_{j=1}^{i-1} Ce^{-|\rho-\xi_j|} + \sum_{i+1}^{k} Ce^{-|\rho-\xi_j|} \\
\leq & \sum_{j=1}^{i-1} Ce^{-\frac{1}{2}(\xi_i-\xi_{i-1})} + \sum_{i+1}^{k} Ce^{-\frac{1}{2}(\xi_{i+1}-\xi_i)} \\
\leq & \, C\varepsilon^{\frac{1}{2}},
\end{aligned}$$

thus

$$\varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \left(V(\rho)^{p+1} - V_i(\rho)^{p+1}\right) d\rho \leq \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} C\varepsilon^{\frac{1}{2}} = C\varepsilon^{\frac{3}{2}}(\zeta_{i+1} - \zeta_i) = o(\varepsilon).$$

Lastly, consider $g(t) = (W_i(\rho) + t\Pi_i(\rho))^{p+1}$. By the Mean value theorem

$$\varepsilon \int_{\zeta_i}^{\zeta i+1} \left(V_i(\rho)^{p+1} - W_i(\rho)^{p+1}\right) d\rho = \varepsilon \int_{\zeta_i}^{\zeta i+1} (p+1)\left(W_i(\rho) + \nu\Pi_i(\rho)\right)^p \Pi_i(\rho)\, d\rho,$$

but $e^{-\xi_i} \leq e^{-\xi_1} \leq C\varepsilon^{\frac{1}{2}}$, then

$$\varepsilon \int_{\zeta_i}^{\zeta i+1} \left(V_i(\rho)^{p+1} - W_i(\rho)^{p+1}\right) d\rho \leq \varepsilon C\varepsilon^{\frac{1}{2}} \int_{\zeta_i}^{\zeta_{i+1}} e^{-\rho}\, d\rho = C\varepsilon^{\frac{3}{2}}(\zeta_{i+1} - \zeta_i) = o(\varepsilon).$$

Therefore, (2.2.8) becomes

$$\varepsilon \int_{\zeta_i}^{\zeta_{i+1}} V_i(\rho)^{p+1} d\rho = \varepsilon \int_{\mathbb{R}} W(\rho)^{p+1} d\rho + o(\varepsilon)$$

and this in turn means that the first integral in (2.2.6) can be estimated as

$$\varepsilon \int_0^\infty V(\rho)^{p+1} d\rho = \varepsilon k \int_{\mathbb{R}} W(\rho)^{p+1} d\rho + o(\varepsilon). \tag{2.2.9}$$

To estimate the second integral in (2.2.6), note that

$$\begin{aligned}
\varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho V(\rho)^{p+1} d\rho = & \, \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho W_i(\rho)^{p+1} d\rho + \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho \left(V(\rho)^{p+1} - W_i(\rho)^{p+1}\right) d\rho \\
\leq & \, \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho W_i(\rho)^{p+1} d\rho + \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho C\varepsilon^{\frac{1}{2}}\, d\rho \\
= & \, \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho W_i(\rho)^{p+1} d\rho + C\varepsilon^{\frac{3}{2}} \left.\frac{\rho^2}{2}\right|_{\zeta_i}^{\zeta_{i+1}} \\
= & \, \varepsilon \int_{\zeta_i}^{\zeta_{i+1}} \rho W_i(\rho)^{p+1} d\rho + o(\varepsilon).
\end{aligned}$$

Next, we want to estimate the integral above. Note that $\rho W(\rho)_i^{p+1}$ is odd, therefore, using the change of variable $y = \rho - \xi_i$

$$\begin{aligned}
\varepsilon\int_{\zeta_i}^{\zeta_{i+1}} \rho W_i(\rho)^{p+1}d\rho = & \ \varepsilon\int_{-\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} (y+\xi_i)W(y)^{p+1}dy \\
= & \ \varepsilon\int_{-\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_i-\xi_{i-1})} yW(y)^{p+1}dy + \varepsilon\int_{\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} yW(y)^{p+1}dy \\
& + \varepsilon\xi_i\int_{-\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} W(y)^{p+1}dy \\
= & \ \varepsilon\int_{\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} yW(y)^{p+1}dy + \varepsilon\xi_i\int_{-\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} W(y)^{p+1}dy,
\end{aligned}$$

but

$$\begin{aligned}
\varepsilon\int_{\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} yW(y)^{p+1}dy \leq & \ \frac{\varepsilon}{2}(\xi_{i+1}-\xi_i)\int_{\frac{1}{2}(\xi_i-\xi_{i-1})}^{\frac{1}{2}(\xi_{i+1}-\xi_i)} W(y)^{p+1}dy \\
\leq & \ C\frac{\varepsilon}{2}(\xi_{i+1}-\xi_i)\int_{\frac{1}{2}(\xi_i-\xi_{i-1})}^{\infty} e^{-(p+1)y}\,dy \\
= & \ C\frac{\varepsilon}{2}(\xi_{i+1}-\xi_i)e^{-\frac{1}{2}(p+1)(\xi_i-\xi_{i-1})} \\
= & \ C\frac{\varepsilon}{2}(\xi_{i+1}-\xi_i)\varepsilon^{\frac{p+1}{2}}\mu_i^{\frac{p+1}{2}} \\
= & \ o(\varepsilon).
\end{aligned}$$

Then,

$$\varepsilon\int_{\zeta_i}^{\zeta_{i+1}} \rho V(\rho)^{p+1}d\rho = \varepsilon\xi_i\int_{\zeta_i}^{\zeta_{i+1}} W(\rho)^{p+1}d\rho + o(\varepsilon) = \varepsilon\xi_i\int_{\mathbb{R}} W(\rho)^{p+1}d\rho + o(\varepsilon)$$

and this in turn implies

$$\varepsilon\int_0^\infty \rho V(\rho)^{p+1}d\rho = \varepsilon\sum_{i=1}^k \xi_i\int_{\mathbb{R}} W(\rho)^{p+1}d\rho + o(\varepsilon), \tag{2.2.10}$$

where

$$\begin{aligned}
\sum_{l=1}^k \xi_i = & \ \xi_1+\xi_2+\xi_3+\xi_4+\dots \\
= & \ -\frac{1}{2}\log\varepsilon + \log\mu_1 - \frac{3}{2}\log\varepsilon - \log\mu_2 + \log\mu_1 \\
& -\frac{5}{2}\log\varepsilon - \log\mu_3 - \log\mu_2 + \log\mu_1 \\
& -\frac{7}{2}\log\varepsilon - \log\mu_4 - \log\mu_3 - \log\mu_2 + \log\mu_1 + \dots \\
= & \ -\frac{k^2}{2}\log\varepsilon + k\log\mu_1 - \sum_{i=2}^k (k-i+1)\log\mu_i.
\end{aligned}$$

Following is to estimate the third integral in (2.2.6). Let

$$\begin{aligned}\varepsilon\int_{\zeta_i}^{\zeta_{i+1}} V(\rho)^{p+1}\log(V(\rho))\,d\rho = {}& \varepsilon\int_{\zeta_i}^{\zeta_{i+1}} W_i(\rho)^{p+1}\log(W_i(\rho))\,d\rho \\ &+\varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V(\rho)^{p+1}\log(V(\rho)) - W_i(\rho)^{p+1}\log(W_i(\rho))\right)d\rho\end{aligned}$$

and define

$$f(t) = W_i(\rho) + t\left(\Pi_i(\rho) + \sum_{j=1,j\neq i}^{k} V_j(\rho)\right) \quad \text{and} \;\; g(t) = (f(t))^{p+1}\log(f(t)).$$

By the Mean value theorem,

$$\varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V(\rho)^{p+1}\log(V(\rho)) - W_i(\rho)^{p+1}\log(W_i(\rho))\right)d\rho = \varepsilon\int_{\zeta_i}^{\zeta_{i+1}}(g(1)-g(0))\,d\rho = \varepsilon\int_{\zeta_i}^{\zeta{i+1}} g'(\nu)\,d\rho,$$

for some $\nu\in(0,1)$ and where $g'(t) = f(t)^p\left((p+1)\log(f(t))+1\right)f'(t)$. Notice that $f(t)^p\left((p+1)\log(f(t))+1\right)\leq C$ as W,Π and V are bounded no matter the choice of ρ. Moreover,

$$f'(t) = \Pi_i(\rho) + \sum_{j=1,j\neq i}^{k} V_j(\rho) \leq C\varepsilon^{\frac{1}{2}} + C\varepsilon^{\frac{1}{2}} = C\varepsilon^{\frac{1}{2}}.$$

Thus,

$$\varepsilon\int_{\zeta_i}^{\zeta_{i+1}}\left(V(\rho)^{p+1}\log(V(\rho)) - W_i(\rho)^{p+1}\log(W_i(\rho))\right)d\rho = C\varepsilon^{\frac{3}{2}} = o(\varepsilon)$$

and this in turn means

$$\varepsilon\int_0^\infty V(\rho)^{p+1}\log(V(\rho))\,d\rho = \varepsilon k\int_{\mathbb{R}} W(\rho)^{p+1}\log(W(\rho))\,d\rho + o(\varepsilon). \tag{2.2.11}$$

Henceforth,

$$\begin{aligned}I_\varepsilon(V) = {}& I_0(V) + \frac{\varepsilon k}{(p+1)^2}\int_{\mathbb{R}} W(\rho)^{p+1}d\rho - \frac{\varepsilon}{p+1}\sum_{i=1}^{k}\xi_i\int_{\mathbb{R}} W(\rho)^{p+1}d\rho \\ &-\frac{\varepsilon k}{p+1}\int_{\mathbb{R}} W(\rho)^{p+1}\log(W(\rho))\,d\rho + o(\varepsilon).\end{aligned} \tag{2.2.12}$$

To continue, we want to estimate $I_0(V)$. For that need to analyze the following

$$\begin{aligned}I_0(V) - \sum_{i=1}^{k} I_0(V_i) = {}& \frac{1}{2}\int_0^\infty\left(V''^2 + C_4V'^2 + C_5V^2\right)d\rho + \frac{2}{N-4}V'(0)^2 - \frac{1}{p+1}\int_0^\infty V^{p+1}d\rho \\ &-\sum_{i=1}^{k}\frac{1}{2}\int_0^\infty\left(V_i''^2 + C_4V_i'^2 + C_5V_i^2\right)d\rho - \frac{2}{N-4}\sum_{i=1}^{k}V_i'(0)^2 \\ &+\frac{1}{p+1}\sum_{i=1}^{k}\int_0^\infty V_i^{p+1}d\rho\end{aligned}$$

$$
\begin{aligned}
= \; & \frac{1}{2}\int_0^\infty \left(V''^2 - \sum_{i=1}^k V_i''^2\right) d\rho + \frac{C_4}{2}\int_0^\infty \left(V'^2 - \sum_{i=1}^k V_i'^2\right) d\rho \\
& + \frac{C_5}{2}\int_0^\infty \left(V^2 - \sum_{i=1}^k V_i^2\right) d\rho + \frac{2}{N-4}\left(V'(0)^2 - \sum_{i=1}^k V_i'(0)^2\right) \\
& - \frac{1}{p+1}\int_0^\infty \left(V^{p+1} - \sum_{i=1}^k V_i^{p+1}\right) d\rho.
\end{aligned}
$$

Notice that

$$
\frac{1}{2}\int_0^\infty \left(V''^2 - \sum_{i=1}^k V_i''^2\right) d\rho = \frac{1}{2}\int_0^\infty 2\sum_{i=2}^k \sum_{j=1}^{i-1} V_i'' V_j'' d\rho = \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty V_i'' V_j''\, d\rho,
$$

$$
\frac{C_4}{2}\int_0^\infty \left(V'^2 - \sum_{i=1}^k V_i'^2\right) d\rho = C_4 \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty V_i' V_j'\, d\rho,
$$

$$
\frac{C_5}{2}\int_0^\infty \left(V^2 - \sum_{i=1}^k V_i^2\right) d\rho = C_5 \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty V_i V_j\, d\rho,
$$

and

$$
\frac{2}{N-4}\left(V'(0)^2 - \sum_{i=1}^k V_i'(0)^2\right) = \frac{4}{N-4}\sum_{i=2}^k \sum_{j=1}^{i-1} V_i'(0)V_j'(0).
$$

Thus, by integrating by part

$$
\begin{aligned}
I_0(V) - \sum_{i=1}^k I_0(V_i) = \; & \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty \left(V_i''V_j'' + C_4 V_i'V_j' + C_5 V_i V_j\right) d\rho + \frac{4}{N-4}\sum_{i=2}^k \sum_{j=1}^{i-1} V_i'(0)V_j'(0) \\
& - \frac{1}{p+1}\int_0^\infty \left(V^{p+1} - \sum_{i=1}^k V_i^{p+1}\right) d\rho \\
= \; & \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty \left(V_i^{(4)} - C_4 V_i'' + C_5 V_i\right) V_j\, d\rho - \frac{1}{p+1}\int_0^\infty \left(V^{p+1} - \sum_{i=1}^k V_i^{p+1}\right) d\rho \\
= \; & \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty W_i^p V_j\, d\rho - \frac{1}{p+1}\int_0^\infty \left(V^{p+1} - \sum_{i=1}^k V_i^{p+1}\right) d\rho \\
= \; & \sum_{i=2}^k \sum_{j=1}^{i-1} \int_0^\infty W_i^p V_j\, d\rho + \frac{1}{p+1}\sum_{l=1}^k \int_{\zeta_l}^{\zeta_{l+1}} \left(V_l^{p+1} - V^{p+1}\right) d\rho \\
& + \frac{1}{p+1}\sum_{l=1}^k \int_{\zeta_l}^{\zeta_{l+1}} \sum_{i=1, i\neq l}^k V_i^{p+1} d\rho.
\end{aligned}
$$

Notice that

$$\begin{aligned}
\sum_{i=2}^{k}\sum_{j=1}^{i-1}\int_0^\infty W_i^pV_j\,d\rho = &\sum_{l=1}^{k}\sum_{i=2}^{k}\sum_{j=1}^{i-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^pV_j\,d\rho\\
= &\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} W_l^pV_j\,d\rho+\sum_{l=1}^{k}\sum_{i=2,i\neq l}^{k}\sum_{j=1}^{i-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^pV_j\,d\rho\\
= &\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} V_l^pV_j\,d\rho+\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} \left(W_l^p-V_l^p\right)V_j\,d\rho\\
&+\sum_{l=1}^{k}\sum_{i=2,i\neq l}^{k}\sum_{j=1}^{i-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^pV_j\,d\rho\\
= &\sum_{l=2}^{k}\sum_{j=1,j\neq l}^{k}\int_{\zeta_l}^{\zeta_{l+1}} V_l^pV_j\,d\rho-\sum_{l=2}^{k-1}\sum_{j=l+1}^{k}\int_{\zeta_l}^{\zeta_{l+1}} V_l^pV_j\,d\rho\\
&+\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} \left(W_l^p-V_l^p\right)V_j\,d\rho+\sum_{l=1}^{k}\sum_{i=2,i\neq l}^{k}\sum_{j=1}^{i-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^pV_j\,d\rho,
\end{aligned}$$

as a consequence

$$\begin{aligned}
I_0(V)-\sum_{i=1}^{k}I_0(V_i) = &\frac{1}{p+1}\sum_{l=1}^{k}\int_{\zeta_l}^{\zeta_{l+1}}\left(V_l^{p+1}-V^{p+1}+(p+1)\sum_{j=1,j\neq l}^{k}V_l^pV_j\right)d\rho\\
&-\sum_{l=1}^{k-1}\sum_{j=l+1}^{k}\int_{\zeta_l}^{\zeta_{l+1}} V_l^pV_jd\rho+\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} \left(W_l^p-V_l^p\right)V_jd\rho\\
&+\sum_{l=1}^{k}\sum_{i=2,i\neq l}^{k}\sum_{j=1}^{i-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^pV_jd\rho+\frac{1}{p+1}\sum_{l=1}^{k}\int_{\zeta_l}^{\zeta_{l+1}}\sum_{i=1,i\neq l}^{k}V_i^{p+1}d\rho.
\end{aligned}$$

Let $B_1, B_2.B_3, B_4$ and B_5 be the terms of the previous equation respectively. We will prove that every term but B_2 are of lower order.

Note that we can easily estimate B_3, B_4 and B_5 using the definition and properties of V_i, W_i and Π_i. Indeed, by the Mean value Theorem

$$\begin{aligned}
B_3 = &\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} \left(W_l^p-V_l^p\right)V_j\,d\rho\\
\leq &\; C\sum_{l=2}^{k}\sum_{j=1}^{l-1}\int_{\zeta_l}^{\zeta_{l+1}} \Pi_lV_j\,d\rho\\
\leq &\; Ce^{-\xi_l}\leq C\varepsilon^{\frac{3}{2}}.
\end{aligned}$$

For B_4, for all $l \geq 2$,

$$\begin{aligned}
B_4 = & \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l}^{\zeta_{l+1}} W_i^p V_j \, d\rho \\
\leq & \; C \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l}^{\zeta_{l+1}} e^{-p|\rho-\xi_i|} e^{-|\rho-\xi_j|} \, d\rho \\
= & \; C \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-p|\rho|} e^{-|\rho+\xi_i-\xi_j|} \, d\rho \\
\leq & \; C \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-p|\rho|} e^{|\rho|-|\xi_i-\xi_j|} \, d\rho \\
= & \; C e^{-|\xi_i-\xi_j|} \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-(p-1)|\rho|} \, d\rho \\
\leq & \; C\varepsilon \sum_{l=1}^{k} \sum_{i=2, i\neq l}^{k} \sum_{j=1}^{i-1} \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p-1)|\rho-\xi_i|} \, d\rho \\
\leq & \; C\varepsilon^{\frac{p+1}{2}}.
\end{aligned}$$

Finally, for B_5,

$$\begin{aligned}
B_5 = & \; \frac{1}{p+1} \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \sum_{i=1, i\neq l}^{k} V_i^{p+1} d\rho \\
\leq & \; \frac{1}{p+1} \sum_{l=1}^{k} \sum_{i=1, i\neq l}^{k} \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p+1)|\rho-\xi_i|} \, d\rho \\
= & \; - \frac{1}{p+1} e^{-(p+1)(\rho-\xi_i)} \Big|_{\zeta_l}^{\zeta_{l+1}} \\
\leq & \; \frac{1}{p+1} e^{-(p+1)(\zeta_l-\xi_i)} \\
\leq & \; C e^{-\frac{p+1}{2}(\xi_{i+1}-\xi_i)} \\
\leq & \; C\varepsilon^{\frac{p+1}{2}}.
\end{aligned}$$

For B_1, define $f(t) = (V_l + t \sum_{j=1, j\neq l}^{k} V_j)^{p+1}$. Using the Mean value Theorem

$$\begin{aligned}
B_1 = & \; \frac{1}{p+1} \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \left(V_l^{p+1} - V^{p+1} + (p+1) \sum_{j=1, j\neq l}^{k} V_l^p V_j \right) d\rho \\
= & \; \frac{1}{p+1} \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \left(f(0) - f(1) + (p+1) \sum_{j=1, j\neq l}^{k} V_l^p V_j \right) d\rho
\end{aligned}$$

$$= \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big(-(V_l + \vartheta \sum_{j=1,j\neq l}^{k} V_j)^p + V_l^p \Big)\Big)\, d\rho \qquad \text{with } \vartheta \in]0,1[.$$

Now, let $g(t) = (V_l + t\vartheta \sum_{j=1,j\neq l}^{k} V_j)^p$. Then

$$\begin{aligned}
B_1 &= \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big(-(V_l + \vartheta \sum_{j=1,j\neq l}^{k} V_j)^p + V_l^p \Big)\Big)\, d\rho \qquad \text{with } \vartheta \in]0,1[\\
&= \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} (-1) \sum_{j=1,j\neq l}^{k} V_j \left(g(0) - g(1) \right)\, d\rho \\
&= p \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1,j\neq l}^{k} V_j \Big(V_l + \tau\vartheta \sum_{j=1,j\neq l}^{k} V_j \Big)^{p-1} \vartheta \sum_{j=1,j\neq l}^{k} V_j \, d\rho \qquad \text{with } \vartheta, \tau \in]0,1[\\
&\leq C \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big)^2 \Big(\sum_{j=1}^{k} V_j \Big)^{p-1} d\rho \\
&\leq C \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p-1)|\rho-\xi_l|} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big)^2 d\rho \\
&= C \sum_{l=1}^{k} \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p-1)|\rho-\xi_l|} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big)^{\frac{p-1}{p}} \Big(\sum_{j=1,j\neq l}^{k} V_j \Big)^{\frac{p+1}{p}} d\rho \\
&\leq C \sum_{l=1}^{k} \left(\int_{\zeta_l}^{\zeta_{l+1}} e^{-p|\rho-\xi_l|} \sum_{j=1,j\neq l}^{k} V_j \, d\rho \right)^{\frac{p-1}{p}} \left(\int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1,j\neq l}^{k} V_j \, d\rho \right)^{\frac{p+1}{p}} \\
&\leq C \sum_{l=1}^{k} \left(\int_{\zeta_l}^{\zeta_{l+1}} e^{-p|\rho-\xi_l|} \sum_{j=1,j\neq l}^{k} e^{-|\rho-\xi_j|} \, d\rho \right)^{\frac{p-1}{p}} \left(\int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1,j\neq l}^{k} V_j^{p+1} d\rho \right)^{\frac{1}{p}} \\
&\leq C \sum_{l=1}^{k} \left(\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} e^{-p|\rho|} \sum_{j=1,j\neq l}^{k} e^{|\rho|-|\xi_l-\xi_j|} \, d\rho \right)^{\frac{p-1}{p}} \varepsilon^{\frac{p+1}{2p}} \\
&\leq C \left(\varepsilon \int_{\mathbb{R}} e^{-(p-1)|\rho|} \, d\rho \right)^{\frac{p-1}{p}} \varepsilon^{\frac{p+1}{2p}} \\
&= C\varepsilon^{\frac{p-1}{p}} \varepsilon^{\frac{p+1}{2p}} \\
&= C\varepsilon^{\frac{3p-1}{2p}}.
\end{aligned}$$

On the other hand, to estimate B_2, we need to separate the cases $j \geq l+2$ and $j = l+1$. The first case implies that $-|\xi_l - \xi_j| \leq 2\log(\varepsilon) - C$ with $C \geq 0$. Then

$$\begin{aligned}
B_2 &= \sum_{l=1}^{k-1}\sum_{j=l+2}^{k}\int_{\zeta_l}^{\zeta_{l+1}} V_l^p V_j \, d\rho \\
&\leq C\sum_{l=1}^{k-1}\sum_{j=l+2}^{k}\int_{\zeta_l}^{\zeta_{l+1}} e^{-p|\rho-\xi_l|}e^{-|\rho-\xi_j|}\, d\rho \\
&\leq C\sum_{l=1}^{k-1}\sum_{j=l+2}^{k}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} e^{-p|\rho|}e^{|\rho|-|\xi_l-\xi_j|}\, d\rho \\
&\leq C\sum_{l=1}^{k-1}\sum_{j=l+2}^{k} e^{-|\xi_l-\xi_j|}\int_{\mathbb{R}} e^{-(p-1)|\rho|}\, d\rho \\
&\leq C\varepsilon^2
\end{aligned}$$

and such, the only relevant term is the case $j = l + 1$. To obtain the correct estimation, we will use the idea in B_3 and (2.2.1) so B_2 can be estimate as

$$\begin{aligned}
B_2 &= \sum_{l=1}^{k-1}\int_{\zeta_l}^{\zeta_{l+1}} V_l^p V_{l+1}\, d\rho \\
&= \sum_{l=1}^{k-1}\int_{\zeta_l}^{\zeta_{l+1}} W_l^p W_{l+1}\, d\rho + o(\varepsilon) \\
&= K_N\sum_{l=1}^{k-1}\int_{\zeta_l}^{\zeta_{l+1}} W_l^p e^{-|\rho-\xi_{l+1}|}\, d\rho + o(\varepsilon) \qquad (2.2.13) \\
&= K_N\sum_{l=1}^{k-1} e^{-(\xi_{l+1}-\xi_l)}\int_{\mathbb{R}} e^{|\rho|}W^p\, d\rho + o(\varepsilon) \\
&= \varepsilon K_N\sum_{i=2}^{k}\mu_i\int_{\mathbb{R}} e^{|\rho|}W^p\, d\rho + o(\varepsilon).
\end{aligned}$$

To recapitulate, what we have is

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = -\varepsilon K_N\sum_{i=2}^{k}\mu_i\int_{\mathbb{R}} e^{|\rho|}W^p d\rho + o(\varepsilon),$$

hence our next step is to estimate $I_0(V_i)$ for all $i = 1, \dots, k$. Consider,

$$I_0(V_i) = I_0(W_i + \Pi_i) = I_0(W_i) + DI_0(W_i)[\Pi_i] + o(\varepsilon).$$

If $i = 1$, by direct calculations and properties of W_1 and Π_1, we have

$$\begin{aligned}
DI_0(W_1)[\Pi_1] = & \int_0^\infty (W_1''(\rho)\Pi_1''(\rho) + C_4 W_1'(\rho)\Pi_1'(\rho) + C_5 W_1(\rho)\Pi_1(\rho))\, d\rho \\
& + \frac{4}{N-4} W_1'(0)\Pi_1'(0) - \int_0^\infty W_1^p(\rho)\Pi_1(\rho)\, d\rho \\
= & -W_1''(0)\Pi_1'(0) + W_1'''(0)\Pi_1(0) - C_4 W_1'(0)\Pi_1(0) + \frac{4}{N-4} W_1'(0)\Pi_1'(0) \\
= & -\frac{N-4}{N} W_1^2(0) - W_1^2(0) + C_5 W_1^2(0) + \frac{4}{N} W_1^2(0) + o(\varepsilon) \\
= & \tfrac{16(N^2-5N+8)}{N(N-4)^2} W_1^2(0) + o(\varepsilon).
\end{aligned}$$

If $i \geq 2$, the result is the same, albeit $W_i^2(0)$ is of lower order. In other words, $W_i^2(0) = o(\varepsilon)$ for all $i \geq 2$. Thus,

$$\begin{aligned}
\sum_{i=1}^k I_0(V_i) = & \sum_{i=1}^k I_0(W_i) + \sum_{i=1}^k DI_0(V_i) + o(\varepsilon) \\
= & \frac{1}{2} \sum_{i=1}^k \Bigg(\int_0^\infty (W_i''(\rho)^2 + C_4 W_i'(\rho)^2 + C_5 W_i(\rho)^2)\, d\rho + \frac{4}{N-4} W_i'(0)^2 \\
& - \frac{1}{p+1} \int_0^\infty W_i^{p+1}(\rho)\, d\rho \Bigg) + \frac{16(N^2-5N+8)}{N(N-4)^2} W_1^2(0) + o(\varepsilon) \\
= & \frac{1}{2} \sum_{i=1}^k \Bigg(\int_{\mathbb{R}} (W''(\rho)^2 + C_4 W'(\rho)^2 + C_5 W(\rho)^2)\, d\rho - \frac{1}{p+1} \int_{\mathbb{R}} W^{p+1}(\rho)\, d\rho \Bigg) \qquad (2.2.14) \\
& + \frac{4}{N-4} W_1'(0)^2 + \frac{16(N^2-5N+8)}{N(N-4)^2} W_1^2(0) + o(\varepsilon) \\
= & \frac{1}{2} \sum_{i=1}^k \Bigg(\int_{\mathbb{R}} (W''(\rho)^2 + C_4 W'(\rho)^2 + C_5 W(\rho)^2)\, d\rho - \frac{1}{p+1} \int_{\mathbb{R}} W^{p+1}(\rho)\, d\rho \Bigg) \\
& + \frac{4(5N^2-24N+32)}{N(N-4)^2} K_N^2 e^{-2\xi_1} + o(\varepsilon).
\end{aligned}$$

Lastly, we need to estimate the functional $K_\varepsilon(V)$. Notice that

$$\begin{aligned}
\lambda_\varepsilon a_\varepsilon \frac{1}{2} \int_0^\infty e^{-\frac{8}{N-4}\rho} V(\rho)^2\, d\rho = & \lambda_\varepsilon a_\varepsilon \frac{1}{2} \sum_{l=1}^k \int_{\zeta_l}^{\zeta_{l+1}} e^{-\frac{8}{N-4}\rho} V_l(\rho)^2\, d\rho + o(\varepsilon) \\
= & \lambda_\varepsilon a_\varepsilon \frac{1}{2} \sum_{l=1}^k \int_{\zeta_l}^{\zeta_{l+1}} e^{-\frac{8}{N-4}\rho} W_l(\rho)^2\, d\rho + o(\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&= \lambda_\varepsilon a_\varepsilon \frac{1}{2} \sum_{i=1}^{k} \int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} e^{-\frac{8}{N-4}(\rho+\xi_l)} W(\rho)^2 \, d\rho + o(\varepsilon) \\
&= \sum_{i=1}^{k} e^{-\frac{8}{N-4}\xi_l} \int_{\mathbb{R}} e^{-\frac{8}{N-4}\rho} W(\rho)^2 \, d\rho + o(\varepsilon) \qquad (2.2.15) \\
&= e^{-\frac{8}{N-4}\xi_1} \int_{\mathbb{R}} e^{-\frac{8}{N-4}\rho} W(\rho)^2 \, d\rho + o(\varepsilon).
\end{aligned}
$$

Therefore, using (2.2.12), (2.2.13), (2.2.14) and (2.2.15), we can expand the functional $E_\varepsilon(V)$ as follow

$$
\begin{aligned}
E_\varepsilon(V) = \; & k a_1 + \varepsilon a_2 \left(\sum_{i=2}^{k} (k-i+1) \log \mu_i - k \log \mu_1 + \tfrac{k^2}{2} (\log \varepsilon) \right) - \varepsilon a_3 \sum_{i=2}^{k} \mu_i + k \varepsilon a_4 + \varepsilon a_5 \tfrac{1}{\mu_1^2} \\
& - \varepsilon a_6 \eta \mu_1^{-\frac{8}{N-4}} + o(\varepsilon),
\end{aligned}
$$

where

$$
a_1 = \frac{1}{2} \sum_{i=1}^{k} \left(\int_{\mathbb{R}} (W''(\rho)^2 + C_4 W'(\rho)^2 + C_5 W(\rho)^2) \, d\rho - \frac{1}{p+1} \int_{\mathbb{R}} W^{p+1}(\rho) \, d\rho \right)
$$

$$
a_2 = \frac{1}{p+1} \int_{\mathbb{R}} W(\rho)^{p+1} d\rho
$$

$$
a_3 = K_N \int_{\mathbb{R}} e^{|\rho|} W^p \, d\rho
$$

$$
a_4 = \frac{1}{(p+1)^2} \int_{\mathbb{R}} W(\rho)^{p+1} d\rho - \frac{1}{p+1} \int_{\mathbb{R}} W(\rho)^{p+1} \log(W(\rho)) \, d\rho
$$

$$
a_5 = K_N^2 \frac{4(5N^2 - 24N + 32)}{N(N-4)^2}
$$

$$
a_6 = \frac{1}{2} a_\varepsilon \int_{\mathbb{R}} e^{-\frac{8}{N-4}\rho} W(\rho)^2 \, d\rho
$$

□

2.3 The Linear Problem

In this section, for $i = 1, \ldots, k$ and $\xi_i \in \mathbb{R}^+$, we will use once again the following notation

$$W_i := W_{\xi_i}, \quad \Pi_i := \Pi_{\xi_i}, \quad V_i := W_i + \Pi_i, \quad \text{and} \quad V = \sum_{i=1}^{k} V_i, \tag{2.3.1}$$

so we can write $v_* = V + \phi$ over $\mathbb{R}^N \setminus B_1$. We will also consider the following functions

$$Z_i := -\frac{\partial V_i}{\partial \xi_i} = -\frac{\partial W_i}{\partial \xi_i} - \frac{\partial \Pi_i}{\xi_i},$$

which will be vital in this section. Moreover, from (2.1.11), this function also satisfies $Z_i(0) = 0$ and $(N-4)Z_i''(0) - 4Z_i'(0) = 0$.

The objective of this section is to obtain the tools necessary to resolve the following problem: *Let* $\boldsymbol{\xi} := (\xi_1, \xi_2, \ldots, \xi_k) \in \mathbb{R}^k$, *we are looking for a function* $\phi : \mathbb{R}^N \setminus B_1 \to \mathbb{R}$ *such that, for some constants* c_i *depending only on* ξ_i, *then* $v_* = V + \phi$ *is the solution of*

$$\begin{cases} L(v_*) = e^{\varepsilon\rho} v_*^{p+\varepsilon}(\rho) + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} v_*(\rho) + \sum\limits_{i=1}^{k} c_i Z_i, \\ \phi(0) = \lim\limits_{\rho\to\infty} \phi(\rho) = 0, \\ (N-4)\phi''(0) - 4\phi'(0) = 0, \\ \int_0^\infty Z_i \phi \, d\rho = 0. \end{cases} \tag{2.3.2}$$

Instead of solving (2.3.2) directly, we consider an intermediate problem. Note that (2.3.2) can be rewritten as

$$\begin{cases} L_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \sum\limits_{i=1}^{k} c_i Z_i, \\ \phi(0) = \lim\limits_{\rho\to\infty} \phi(\rho) = 0, \\ (N-4)\phi''(0) - 4\phi'(0) = 0, \\ \int_0^\infty Z_i \phi \, d\rho = 0, \end{cases} \tag{2.3.3}$$

where

$$L_\varepsilon(\phi) = L(\phi) - e^{\varepsilon\rho}(p+\varepsilon)V^{p+\varepsilon-1}\phi - \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho}\phi, \tag{2.3.4}$$

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}\left((V+\phi)^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi\right) \tag{2.3.5}$$

and

$$R_\varepsilon = e^{\varepsilon\rho}V^{p+\varepsilon} - \sum_{i=1}^{k} W_i^p + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} V. \tag{2.3.6}$$

To continue, it is convenient to introduce the following norm. Given a fixed and small $\sigma > 0$, define

$$\|g\|_* := \sup_{\rho\in\mathbb{R}^+} \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} |g(\rho)|.$$

Also, we consider the Banach space

$$\mathcal{C}^*((0,\infty)) = \{f \in C((0,\infty)) : \|f\|_* < \infty\}$$

and the linear space $\mathcal{L}(\mathcal{C}^*((0,\infty)))$. Then, given $h \in \mathcal{L}(\mathcal{C}^*((0,\infty)))$ such that

$$\int_0^\infty hZ_i\, d\rho = 0 \quad \text{for all } i = 1,\dots,k,, \tag{2.3.7}$$

the objective is to find ϕ such that, for some c_i, it satisfies the problem

$$\begin{cases} L_\varepsilon(\phi) = & h + \sum\limits_{i=1}^{k} c_i Z_i, \\ \phi(0) = \lim\limits_{\rho\to\infty} \phi(\rho) = & 0, \\ (N-4)\phi''(0) - 4\phi'(0) = & 0, \\ \int_0^\infty Z_i \phi\, d\rho = & 0. \end{cases} \tag{2.3.8}$$

First, we need to prove the following lemma.

Lemma 2.3.1. *Let $\{\varepsilon_n\}_{n\in\mathbb{N}}$ such that $\varepsilon_n \to 0$. Consider $0 < \xi_1^n < \cdots < \xi_k^n$ depending on ε_n satisfying*

$$\xi_1^n \to \infty, \qquad \min_{i=1,\dots,k-1} (\xi_{i+1}^n - \xi_i^n) \to \infty. \tag{2.3.9}$$

Assume that there is $\{\phi_n\}_{n\in\mathbb{N}}$ solutions of (2.3.8) for $h_n \in \mathcal{C}^*((0,\infty))$, h_n satisfies (2.3.7) and $\|h_n\|_* \to 0$. Then, $\|\phi_n\|_* \to 0$.

Proof. First, we will prove that $\lim\limits_{n\to\infty} \|\phi\|_\infty \to 0$. By contradiction, take $\|\phi_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Then, integrating (2.3.8) against $Z_{l,n}$, we have

$$\begin{aligned} \int_0^\infty h_n Z_{l,n}\, d\rho\ + \sum_0^\infty c_{i,n} Z_{i,n} Z_{l,n}\, d\rho &= \int_0^\infty L_{\varepsilon_n}(\phi_n) Z_{l,n}\, d\rho \\ &= \int_0^\infty \Big(\phi_n^{(4)} - C_4\phi_n'' + C_5\phi_n - (p+\varepsilon_n)e^{\varepsilon_n\rho} V^{p+\varepsilon_n-1}\phi_n - \lambda_{\varepsilon_n} a_{\varepsilon_n} e^{-\frac{8}{N-4}\rho}\phi_n\Big) Z_{l,n}\, d\rho \\ &= \phi_n''' Z_{l,n}\big|_0^\infty - \phi_n'' Z_{l,n}'\big|_0^\infty + \phi_n' Z_{l,n}''\big|_0^\infty - \phi_n Z_{l,n}'''\big|_0^\infty - C_4\phi_n' Z_{l,n}\big|_0^\infty + C_4\phi_n Z_{l,n}'\big|_0^\infty \\ &\quad + \int_0^\infty L_{\varepsilon_n}(Z_{l,n})\phi_n\, d\rho \\ &= \int_0^\infty L_{\varepsilon_n}(Z_{l,n})\phi_n\, d\rho. \end{aligned}$$

Note that, from calculations performed in the previous chapter and by the definition of Z_i, we have

$$\sum_{i=1}^{k}\int_0^\infty c_{i,n}Z_{i,n}Z_{l,n}\,d\rho = \begin{cases} 0 & \text{if } i \neq l, \\ a_i & \text{if } i = l, \end{cases}$$

when $n \to \infty$. Moreover, as $\|h_n\|_* \to 0$ if $n \to \infty$, there is $\Theta_n(\rho)$ such that

$$|h_n(\rho)| \leq \Theta_n(\rho)\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}$$

and $\Theta_n \to 0$ uniformly as $n \to \infty$.

Even more, knowing that $Z_{l,n}(\rho) = O(e^{-|\rho-\xi_l^n|})$ due to Proposition 2.2.1, the following inequality follows

$$\left|\int_0^\infty h_n Z_{l,n}\,d\rho\right| \leq C\|\Theta_n(\rho)\|_\infty \left|\int_0^\infty e^{-(1+\sigma)|\rho-\xi_l|}\,d\rho\right| \to 0,$$

if $n \to \infty$. On the other hand, notice that

$$W_{l,n}^{(4)}(\rho) - C_4 W_{l,n}''(\rho) + C_5 W_{l,n}(\rho) = W_{l,n}^p(\rho),$$

then, as $Z_{l,n} = -\frac{\partial W_{l,n}}{\partial \xi_l}(\rho) - \frac{\partial \Pi_{l,n}}{\partial \xi_l}(\rho)$, we have

$$\begin{aligned} L(Z_{l,n}) =\ & Z_{l,n}^{(4)}(\rho) - C_4 Z_{l,n}''(\rho) + C_5 Z_{l,n}(\rho) \\ =\ & -\left(\frac{\partial W_{l,n}^{(4)}}{\partial \xi_l}(\rho) - C_4\frac{\partial W_{l,n}''}{\partial \xi_l}(\rho) + C_5\frac{\partial W_{l,n}}{\partial \xi_l}(\rho)\right) \\ & -\left(\frac{\partial \Pi_{l,n}^{(4)}}{\partial \xi_l}(\rho) - C_4\frac{\partial \Pi_{l,n}''}{\partial \xi_l}(\rho) + C_5\frac{\partial \Pi_{l,n}}{\partial \xi_l}(\rho)\right) \\ =\ & -\frac{\partial}{\partial \xi_l} W_{l,n}^p(\rho) \\ =\ & pW_{l,n}^{p-1}\left(Z_{l,n} + \tfrac{\partial \Pi_{l,n}}{\partial \xi_l}(\rho)\right), \end{aligned}$$

where $pW_{l,n}^{p-1}\frac{\partial \Pi_{l,n}}{\partial \xi_l}(\rho) \to 0$ when $n \to \infty$. Thus, using Dominated convergence Theorem, we can conclude

$$\int_0^\infty \left(pW_{l,n}^{p-1} - e^{\varepsilon_n\rho}(p+\varepsilon_n)V^{p+\varepsilon_n-1} - \lambda_{\varepsilon_n}a_{\varepsilon_n}e^{-\frac{8}{N-4}\rho}\right) Z_{l,n}\phi_n\,d\rho \to 0,$$

if $n \to \infty$ and this in turn implies $c_{i,n} \to 0$.

Now, choose $\rho_n \in \mathbb{R}^+$ such that $\phi_n(\rho_n) = 1$, in other word, its maximum. We will prove that, for some big enough n, there is $\overline{r} > 0$ and $i \in \{1, \ldots, k\}$ such that

$$|\rho_n - \xi_{i,n}| < \overline{r},$$

when $n \to \infty$.

Assume the contrary, either $|\rho_n| \to \infty$ or $|\rho_n|$ remains bounded. First, let $|\rho_n - \xi_{i,n}| \to \infty$ if $n \to \infty$

for all $i \in \{1, \dots, k\}$. Define

$$\tilde{\phi}_n(\rho) = \phi_n(\rho + \rho_n),$$

then, $\{\tilde{\phi}(\rho)\}_{n\in\mathbb{N}}$ converges uniformly on compact, up to subsequences, to a nontrivial solution $\tilde{\phi}$ of the equation

$$L(\tilde{\phi}) = 0.$$

Taking $\tilde{\psi} = \mathcal{T}^{-1}(\tilde{\phi})$, where

$$\tilde{\psi}(x) = \left(\frac{N-4}{2}\right)^{\frac{N-4}{2}} |x|^{-\frac{N-4}{2}} \tilde{\phi}\left(-\frac{N-4}{2}\log(|x|)\right)$$

and $\tilde{\psi}$ satisfies $-\Delta^2\tilde{\psi} = 0$ in $\mathbb{R}^N \setminus \{0\}$, we can use our previous assumption $\|\tilde{\phi}\|_\infty = 1$ to get

$$|\tilde{\psi}(x)| \leq \left(\frac{N-4}{2}\right)^{\frac{N-4}{2}} |x|^{-\frac{N-4}{2}} \quad \text{in } \mathbb{R}^N \setminus \{0\}$$

and this in turns implies that $\|\tilde{\psi}\|_\infty = 0$, which is a contradiction.
In the same way, if $|\rho_n|$ is bounded, we can define

$$\tilde{\phi}_n(\rho) = \phi_n(\rho + \xi_i^n)$$

and use the same argument in the previous case to reach a contradiction again.

Therefore, the sequence $\{\tilde{\phi}_n\}_{n\in\mathbb{N}}$ converges uniformly on compact, up to subsequences, to a nontrivial solution ϕ of the equation

$$L(\phi) = pW^{p-1}\phi.$$

In this way, we can consider ψ satisfying $\psi = \mathcal{T}^{-1}(\phi)$ such that

$$\psi(x) := \left(\frac{N-4}{2}\right)^{\frac{N-4}{2}} |x|^{\frac{N-4}{2}} \phi\left(-\frac{N-4}{2}\log|x|\right)$$

and this in turn implies that ψ verifies

$$\Delta^2\psi(x) = pU^{p-1}(x)\psi(x).$$

Notice that, by [28, Section 2], ψ must satisfy

$$\psi(x) \in \operatorname{span}\left\{\frac{\partial U_{\mu,y}}{\partial \mu}, \frac{\partial U_{\mu,y}}{\partial y_1}, \frac{\partial U_{\mu,y}}{\partial y_2}, \dots, \frac{\partial U_{\mu,y}}{\partial y_N}\right\},$$

but ψ is a radial solution, thus

$$\begin{aligned}
\psi(x) :=\ & C\tfrac{\partial U_{\mu,0}}{\partial \mu} \\
=\ & C\tfrac{\partial}{\partial \mu}\left(\overline{\gamma}_{N,2}\mu^{\frac{N-4}{2}}(1+|\mu x|^2)^{-\frac{N-4}{2}}\right) \\
=\ & \tfrac{N-4}{2}\overline{\gamma}_{N,2}\mu^{\frac{N-6}{2}}(1+|\mu x|^2)^{-\frac{N-4}{2}} - (N-4)\overline{\gamma}_{N,2}\mu^{\frac{N-2}{2}}(1+|\mu x|^2)^{-\frac{N-2}{2}}|x|^2,
\end{aligned}$$

for some constants $C, \mu \in \mathbb{R}$. Then, by definition and (2.1.20), we have

$$\begin{aligned}
\phi(\rho) = \mathcal{T}[\psi](\rho) = & \left(\tfrac{2}{N-4}\right)^{\frac{N-4}{2}} e^{-\rho}\psi(e^{-\frac{2}{N-4}\rho}) \\
= & \; CK_N \tfrac{N-4}{2}\mu^{\frac{N-6}{2}} e^{-\rho}\left(1+\mu^2 e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-4}{2}} \\
& - CK_N(N-4)\mu^{\frac{N-2}{2}} e^{-\rho}\left(1+\mu^2 e^{-\frac{4}{N-4}\rho}\right)^{-\frac{N-2}{2}} e^{-\frac{N}{N-4}\rho} \\
= & \; C\left(-\tfrac{N-4}{2}\right)\Bigg(-K_N e^{-(\rho-\xi)}\left(1+e^{-\frac{4}{N-4}(\rho-\xi)}\right)^{-\frac{N-4}{2}}\mu^{-1} \\
& + 2K_N e^{-\frac{N}{N-4}(\rho-\xi)}\left(1+e^{-\frac{4}{N-4}(\rho-\xi)}\right)^{-\frac{N-2}{2}}\mu^{-1}\Bigg) \\
= & \; C\mu^{-1}W'(\rho-\xi) \\
= & \; CW'(\rho-\xi).
\end{aligned}$$

To obtain ξ, it is enough to check the following. If $\xi \neq 0$, then

$$\begin{aligned}
pW^{p-1}(\rho)\phi(\rho) = & \; L(\phi) \\
= & \; \phi^{(4)}(\rho) - C_4\phi''(\rho) + C_5\phi(\rho) \\
= & \; C((W'(\rho-\xi))^{(4)} - C_4(W'(\rho-\xi))'' + C_5W'(\rho-\xi)) \\
= & \; C(W^{(4)}(\rho-\xi) - C_4W''(\rho-\xi) + C_5W(\rho-\xi))' \\
= & \; C(W^p(\rho-\xi))' \\
= & \; CpW^{p-1}(\rho-\xi)W'(\rho-\xi) \\
= & \; pW^{p-1}(\rho-\xi)\phi(\rho).
\end{aligned}$$

Choosing $\rho = 0$, we would get

$$pW^{p-1}(0)\phi(0) = pW^{p-1}(-\xi)\phi(0),$$

which is a contradiction, because $W(\rho)$ has an unique maximum at $\rho = 0$. Therefore, $\xi = 0$ and this in turn implies that $\phi(\rho) = CW'(\rho)$.
Finally, using the orthogonality condition in (2.3.8), we can conclude that

$$0 = \int_0^\infty Z_n(\rho - \xi_i^n)\phi_n(\rho)\,d\rho = \int_{-\xi_i^n}^\infty Z_n(\rho)\phi_n(\rho+\xi_i^n)\,d\rho \to C\int_{\mathbb{R}} (W')^2(\rho)\,d\rho,$$

thus $W' \equiv 0$, which is a contradiction. Consequently, $\|\phi_n\|_\infty \to 0$.

To continue, let $g_n := L(\phi_n)$, in other words

$$g_n = h_n + \sum_{i=1}^k c_{i,n}Z_{i,n} + (p+\varepsilon_n)e^{\varepsilon_n\rho}V^{p+\varepsilon_n-1}\phi_n + \lambda_{\varepsilon_n}a_{\varepsilon_n}e^{-\frac{8}{N-4}\rho}\phi_n.$$

Notice that

$$\begin{aligned}
|g_n| \leq \; & |h_n| + \sum_{i=1}^{k} |c_{i,n}||Z_{i,n}| + (p+\varepsilon_n)e^{\varepsilon_n \rho}|V|^{p+\varepsilon_n-1}|\phi_n| + \lambda_{\varepsilon_n} a_{\varepsilon_n} e^{-\frac{8}{N-4}\rho}|\phi_n| \\
\leq \; & \|h_n\|_* \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i^n|} + C c_{\max,n} \sum_{i=1}^{k} e^{-|\rho-\xi_i^n|} + C\|\phi_n\|_\infty \sum_{i=1}^{k} e^{-(p-1)|\rho-\xi_i^n|} \\
& + C\|\phi_n\|_\infty \sum_{i=1}^{k} e^{-\frac{8}{N-4}|\rho-\xi_i^n|} \\
\leq \; & C(\|h_n\|_* + c_{\max,n} + \|\phi_n\|_\infty) \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i^n|},
\end{aligned}$$

with $\sigma \in (0, \min\{1, p-1, \frac{8}{N-4}\})$. Now, knowing that $|x| = e^{-\frac{2}{N-4}\rho}$, define

$$\begin{aligned}
p_n(\rho) :=\; & C(\|h_n\|_* + c_{\max,n} + \|\phi_n\|_\infty) \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i^n|}, \\
q_n(x) :=\; & \left(\tfrac{N-4}{2}\right)^{\frac{N-4}{2}} e^{-\rho} p_n(\rho), \\
r_n(x) :=\; & \left(\tfrac{N-4}{2}\right)^{\frac{N-4}{2}} e^{-\rho} \phi_n(\rho).
\end{aligned}$$

By direct calculations, it is easy to see that

$$\begin{aligned}
\Delta^2 q_n =\; & \left(\tfrac{N-4}{2}\right)^{\frac{N+4}{2}} e^{-p\rho}(\sigma^4 - C_4\sigma^2 + C_5\sigma) p_n(\rho), \\
\Delta^2 r_n \leq\; & \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}} e^{-p\rho} p_n(\rho), \\
\Delta q_n \leq\; & \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}} e^{-\frac{N}{N-4}\rho} \left(\sigma^2 + \tfrac{4}{N-4}\sigma - \tfrac{N}{N-4}\right) p_n(\rho) \leq 0, \\
\Delta r_n =\; & \left(\tfrac{N-4}{2}\right)^{\frac{N}{2}} e^{-\frac{N}{N-4}\rho} \left(\phi_n''(\rho) + \tfrac{4}{N-4}\phi_n'(\rho) - \tfrac{N}{N-4}\phi_n(\rho)\right).
\end{aligned}$$

Also, by (2.3.8), when $-\frac{N-4}{2}\log|x| = \rho = 0$, we have $\Delta r_n = r_n = 0$. Then, after choosing a sufficiently large constant, we have

$$\begin{cases}
\Delta^2(Cq_n) \geq \Delta^2 r_n & \text{in } B_1, \\
\Delta(Cq_n) \leq \Delta r_n & \text{on } \partial B_1, \\
Cq_n \geq r_n & \text{on } \partial B_1.
\end{cases}$$

Now, define $s_n(\rho) = Cq_n(\rho) - r_n(\rho)$ and $w_n(\rho) = \Delta s_n(\rho)$. The previous equation let us write

$$\begin{cases}
\Delta(w_n) \geq 0 & \text{in } B_1, \\
w_n \leq 0 & \text{on } \partial B_1,
\end{cases}$$

thus, we can use the maximum principle to conclude that $w_n(\rho) \leq 0$. Using the same argument, we can write

$$\begin{cases}
\Delta(s_n) \leq 0 & \text{in } B_1, \\
s_n \geq 0 & \text{on } \partial B_1
\end{cases}$$

and then, using the minimum principle, conclude that $s_n(\rho) \geq 0$ and this in turn implies that $Cq_n(\rho) \geq r_n(\rho)$. Likewise, we can also conclude that $-Cq_n(\rho) \leq r_n(\rho)$. Accordingly,

$$\begin{aligned} |\phi_n(\rho)| \leq\ & C(\|h_n\|_* + c_{\max,n} + \|\phi_n\|_\infty) \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i^n|} \\ \|\phi_n\|_* \leq\ & C(\|h_n\|_* + c_{\max,n} + \|\phi_n\|_\infty) \to 0, \end{aligned}$$

finishing the proof. □

With this technical lemma proved, we can proceed to prove the following proposition

Proposition 2.3.2. *There exist positive constants $\overline{\varepsilon}_k$, $\overline{\delta}_k$, $\overline{r}_k$ and $\overline{\lambda}$ such that if $\boldsymbol{\xi} \in \mathcal{M}_\varepsilon$, where*

$$\mathcal{M}_\varepsilon := \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \ \frac{\overline{r}_k}{2} < \xi_1, \quad \overline{r}_k < \min_{1 \leq i \leq k-1} (\xi_{i+1} - \xi_i) \quad y \quad \xi_k < \frac{\overline{\delta}_k}{\varepsilon} \right\},$$

then, for every $\varepsilon \in (0, \overline{\varepsilon_k})$, $\lambda \in (0, \overline{\lambda})$ and for every $h \in \mathcal{C}^*((0,\infty))$ satisfying (2.3.7), the problem (2.3.8) has a unique solution $\phi := T_\varepsilon(h)$. Furthermore, there exists a constant $C > 0$ such that

$$\|T_\varepsilon(h)\|_* \leq C \|h\|_*$$

and

$$|c_i| \leq C \|h\|_* .$$

Proof. Consider the Hilbert space

$$H_\varepsilon = \left\{ \phi \in H^2((0,\infty)) \cap H_0^1((0,\infty)) : \int_0^\infty Z_i \phi \, d\rho = 0 \quad \text{for } i = 1, \ldots, k \right\},$$

endowed with the inner product

$$(w,v)_{H_\varepsilon} := \int_0^\infty (w''(\rho)v''(\rho) + C_4 w'(\rho)v'(\rho) + C_5 w(\rho)v(\rho)) \, d\rho \ + \frac{4}{N-4} w'(0)v'(0).$$

So, the problem (2.3.8) written in its weak form with respect to H_ε is equivalent to finding $\phi \in H_\varepsilon$ such that

$$(\phi,\psi)_{H_\varepsilon} = \int_0^\infty \left((p+\varepsilon) e^{\varepsilon\rho} V^{p+\varepsilon-1} + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} \right) \phi\psi \, d\rho + \int_0^\infty h\psi \, d\rho,$$

for some $\psi \in H_\varepsilon$. As H_ε is Hilbert, by using the Riesz Representation Theorem, a linear isomorphism $\mathcal{F}_\varepsilon \in \mathcal{L}(H_\varepsilon, H_\varepsilon)$ exists, such that a unique $\phi \in H_\varepsilon$ is identified for each $\phi^* \in H_\varepsilon^*$. This identification satisfies $\mathcal{F}_\varepsilon(\phi)\,[\psi] = (\phi,\psi)_{H_\varepsilon}$ for every $\psi \in H_\varepsilon$. Thus, ϕ can be identified with $\mathcal{F}_\varepsilon(\phi^*)$. On the other hand, the operator $M_\varepsilon : H_\varepsilon \to H_\varepsilon^*$ defined by

$$M_\varepsilon(\phi)\,[\psi] = \int_0^\infty \left((p+\varepsilon) e^{\varepsilon\rho} V^{p+\varepsilon-1} + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho} \right) \phi\psi \, d\rho,$$

is compact and the functional $\tilde{h} : H_\varepsilon \to \mathbb{R}$, defined by

$$\psi \mapsto \tilde{h}(\psi) = \int_0^\infty h\psi \, d\rho,$$

clearly belongs to H_ε^*. Therefore, (2.3.8) can be interpreted as an operator on H_ε, and the task is to find $\phi \in H_\varepsilon$ such that

$$\phi := T_\varepsilon(h) = M_\varepsilon(\phi) + \tilde{h}.$$

The Fredholm Alternative Theorem guarantees that this problem has a unique solution for any $h \in H_\varepsilon$ as long as the homogeneous equation $\phi = M_\varepsilon(\phi)$ has only the trivial solution in H_ε. Now, it is observed that in a weak sense in H_ε, solving this equation is equivalent to solving the problem

$$\begin{cases} L_\varepsilon(\phi) = \sum_{i=1}^{k} c_i Z_i, \\ \phi(0) = \lim_{\rho\to\infty} \phi(\rho) = 0, \\ (N-4)\phi''(0) - 4\phi'(0) = 0, \\ \int_0^\infty Z_i \phi \, d\rho = 0, \end{cases} \tag{2.3.10}$$

for some constants c_is. To prove that (2.3.10) has only the trivial solution in H_ε, a proof by contradiction is employed. Let ϕ be a nontrivial solution of (2.3.10). Without loss of generality, it can be assumed that $\|\phi\|_* = 1$. Setting $\phi = \phi_n$ and $h_n = 0$, and considering sequences $\varepsilon_n \to 0$ as $n \to \infty$ and ξ_i^n as in (2.3.9), all conditions are met to apply Lemma 2.3.1. Consequently, it is concluded that

$$\|\phi_n\|_* \to 0 \quad \text{if } n \to \infty,$$

leading to a contradiction.

Therefore, for constants $\bar{\varepsilon}_k$, $\bar{\delta}_k$, $\bar{r}_k$ and $\bar{\lambda}$, it follows that for $0 < \varepsilon < \bar{\varepsilon}_k$, $0 < \lambda < \bar{\lambda}$ and $h \in \mathcal{C}^*((0,\infty))$, the problem (2.3.8) has a unique solution in H_ε.
Next, we will prove that $\phi = T_\varepsilon(h)$ verifies the inequality

$$\|\phi\|_* \le C \|h\|_*,$$

for some constant $C > 0$. Once again, we will proceed by contradiction. Let ϕ be a nontrivial solution of (2.3.8). Without loss of generality, we can assume that $\|\phi\|_* = 1$. Fixing $h = h_n$ and $\phi = \phi_n$ with

$$\|h_n\|_* < \frac{1}{n} \|\phi_n\|_*,$$

then, once again, all the conditions are met to apply Lemma 2.3.1 and it can be concluded that

$$\|\phi_n\|_* \to 0 \quad \text{if } n \to \infty,$$

which is a contradiction.
Lastly, to prove the inequality

$$|c_i| \le C \|h\|_*,$$

it is sufficient to follow the same idea as in Lemma 2.3.1. Indeed,

$$\begin{aligned}
L_\varepsilon(\phi) =&\ h + \sum_{i=1}^{k} c_i Z_i \\
\sum_{i=1}^{k} c_i \int_0^\infty Z_i Z_j d\rho =&\ \int_0^\infty L_\varepsilon(\phi) Z_j d\rho - \int_0^\infty h Z_j d\rho \\
|c_i| \int_0^\infty Z_j^2 d\rho + o(\varepsilon) =&\ \int_0^\infty L_\varepsilon(Z_j)\phi d\rho - \int_0^\infty h Z_j d\rho \\
|c_j| \leq&\ C\left(\left|\int_0^\infty L_\varepsilon(Z_j)\phi d\rho\right| + \left|\int_0^\infty h Z_j d\rho\right|\right) \\
|c_j| \leq&\ C(\|\phi\|_* + \|h\|_*) \\
|c_j| \leq&\ C\|h\|_*.
\end{aligned}$$

□

Another property we are interested in is the differentiability of T_ε in the variables ξ_i. Indeed, it is not difficult to prove the following Proposition

Proposition 2.3.3. *Under the assumptions of Proposition 2.3.2, define* $S_\varepsilon(\boldsymbol{\xi})(h) = T_\varepsilon(h)$. *Then, for each* $h \in \mathcal{C}^*((0,\infty))$, *the mapping* $\boldsymbol{\xi} \mapsto S_\varepsilon(\boldsymbol{\xi}, h)$ *is of class* C^1. *Moreover, there exists a constant* $C > 0$ *such that*

$$\left\|\nabla_{\boldsymbol{\xi}} T_\varepsilon(h)\right\|_* \leq C \left\|h\right\|_*,$$

uniformly on vectors $\boldsymbol{\xi} \in \mathcal{M}_\varepsilon$.

Proof. Let $h \in \mathcal{C}^*((0,\infty))$ and define $\phi = T_\varepsilon(h)$. From Proposition 2.3.2, ϕ satisfies

$$L_\varepsilon(\phi) = h + \sum_{i=1}^{k} c_i Z_i.$$

Let us choose a fixed but arbitrary $l \in \{1,\ldots,k\}$. Then, we can differentiate the equation above with respect to ξ_l, to obtain

$$\begin{aligned}
\frac{\partial}{\partial \xi_l} L_\varepsilon(\phi) =&\ \frac{\partial}{\partial \xi_l}\left(h + \sum_{i=1}^{k} c_i Z_i\right) \\
\frac{\partial}{\partial \xi_l}\left(L(\phi) - e^{\varepsilon\rho}(p+\varepsilon)V^{p+\varepsilon-1}\phi - \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho}\phi\right) =&\ \sum_{i=1}^{k} \frac{\partial c_i}{\partial \xi_l} Z_i + \sum_{i=1}^{k} c_i \frac{\partial Z_i}{\partial \xi_l} \\
L_\varepsilon\left(\frac{\partial \phi}{\partial \xi_l}\right) - e^{\varepsilon\rho}(p+\varepsilon)\frac{\partial V^{p+\varepsilon-1}}{\partial \xi_l}\phi =&\ \sum_{i=1}^{k} \frac{\partial c_i}{\partial \xi_l} Z_i + c_l \frac{\partial Z_l}{\partial \xi_l}.
\end{aligned}$$

The next step is to consider the constants a_i that satisfy

$$\sum_{i=1}^{k} a_i \int_0^\infty Z_i Z_j \, d\rho = \int_0^\infty \phi \frac{\partial Z_j}{\partial \xi_l} \, d\rho, \tag{2.3.11}$$

for every $j \in \{1,\ldots,k\}$, which exist because the associated linear system is diagonally dominant. Define the function r as

$$r := \sum_{i=1}^{k} a_i L_\varepsilon(Z_i) + c_l \frac{\partial Z_l}{\partial \xi_l} + e^{\varepsilon\rho}(p+\varepsilon)\frac{\partial V^{p+\varepsilon-1}}{\partial \xi_l}\phi,$$

thus

$$L_\varepsilon\left(\frac{\partial\phi}{\partial\xi_l}+\sum_{i=1}^{k}a_iZ_i\right)=r+\sum_{i=1}^{k}\frac{\partial c_i}{\partial\xi_l}Z_i.$$

Let $s=\frac{\partial\phi}{\partial\xi_l}+\sum\limits_{i=1}^{k}a_iZ_i$. By direct calculations, we notice that

$$\begin{aligned}\int_0^\infty sZ_j\,d\rho &= \int_0^\infty\frac{\partial\phi}{\partial\xi_l}Z_j+\sum_{i=1}^{k}a_i\int_0^\infty Z_iZ_j\,d\rho\\ &= \int_0^\infty\frac{\partial\phi}{\partial\xi_l}Z_j\,d\rho+\int_0^\infty\phi\frac{\partial Z_j}{\partial\xi_l}\,d\rho\\ &= \int_0^\infty\frac{\partial}{\partial\xi_l}(\phi Z_j)\,d\rho\\ &= 0,\end{aligned}$$

for every $j\in\{1,\ldots,k\}$. Note that, by the previous calculation, s satisfies the orthogonality condition of (2.3.8). Moreover, since both ϕ and Z_j satisfy the boundary conditions of (2.3.8), s also satisfies them. Therefore, by Proposition 2.3.2, we have

$$\begin{aligned}T_\varepsilon(r) &= s\\ T_\varepsilon(r)-\sum_{i=1}^{k}a_iZ_i &= \tfrac{\partial\phi}{\partial\xi_l}.\end{aligned}\tag{2.3.12}$$

Now, notice that if $j=l$ in (2.3.11), then

$$\begin{aligned}\sum_{i=1}^{k}a_i\int_0^\infty Z_iZ_j\,d\rho &= \int_0^\infty\phi\frac{\partial Z_j}{\partial\xi_l}\,d\rho\\ |a_l|\int_0^\infty Z_l^2d\rho &= \left|\int_0^\infty\phi\frac{\partial Z_j}{\partial\xi_l}\,d\rho-\sum_{i=1,i\neq l}^{k}a_i\int_0^\infty Z_iZ_j\,d\rho\right|\\ &\leq \|\phi\|_\infty\int_0^\infty\left|\frac{\partial Z_j}{\partial\xi_l}\right|d\rho+o(\varepsilon)\\ &\leq C\|\phi\|_\infty,\end{aligned}$$

thus, $a_l\leq C\|\phi\|_\infty$. On the other hand, if $j\neq l$, then

$$\begin{aligned}\sum_{i=1}^{k}a_i\int_0^\infty Z_iZ_j\,d\rho &= \int_0^\infty\phi\frac{\partial Z_j}{\partial\xi_l}\,d\rho\\ \sum_{i=1}^{k}a_i\int_0^\infty Z_iZ_j\,d\rho &= 0\\ |a_l|\int_0^\infty Z_l^2\,d\rho &= \left|\sum_{i=1,i\neq l}^{k}a_i\textstyle\int_0^\infty Z_iZ_j\,d\rho\right|\\ &= o(\varepsilon),\end{aligned}$$

thus, $a_j\to0$. Due to the previous calculation, we can use Proposition 2.2.1 and Proposition 2.3.2

to estimate r as

$$\begin{aligned}\|r\|_* &\le \sum_{i=1}^k \|a_i L_\varepsilon(Z_i)\|_* + \|c_l \tfrac{\partial Z_l}{\partial \xi_l}\|_* + C\|\tfrac{\partial V^{p+\varepsilon-1}}{\partial \xi_l}\phi\|_* \\ &\le C\sum_{i=1}^k |a_i| + C|c_l| + C\|\phi\|_* \\ &\le C\|\phi\|_* + C\|h\|_* + C\|h\|_* \\ &\le C\|h\|_*.\end{aligned}$$

Henceforth, we conclude that $\|\frac{\partial \phi}{\partial \xi_l}\|_* \le C\|h\|_*$. Furthermore, since we chose l arbitrarily, we have

$$\begin{aligned}\|\nabla_{\boldsymbol{\xi}} T_\varepsilon(h)\|_* &\le \sum_{i=1}^k \|\frac{\partial T_\varepsilon(h)}{\partial \xi_i}\|_* \\ &\le C\|h\|_*.\end{aligned}$$

□

In the remainder of this section, we will consider ξ_i satisfying the conditions required in the previous lemmas, which we will express for simplicity as

$$\frac{1}{2}\log\frac{1}{M\varepsilon} < \xi_1, \quad \log\frac{1}{M\varepsilon} < \min_{i=2,\dots,k}\{\xi_i - \xi_{i-1}\}, \quad \xi_k < k\log\frac{1}{M\varepsilon} \quad \text{and} \quad \lambda_\varepsilon < M\varepsilon^{\frac{N-8}{N-4}}, \tag{2.3.13}$$

with $M > 0$ fixed and sufficiently large.

Next, we need to estimate $N_\varepsilon(\phi)$ and R_ε, defined in (2.3.5) and (2.3.6) respectively. For that, we announce the following lemma

Lemma 2.3.4. *Let $\xi_1, \xi_2, \dots, \xi_k$ satisfying (2.3.13). Assume additionally that $\|\phi\|_* \le \frac{1}{4}$ and σ is sufficiently small. Then, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$\|N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p+\varepsilon_0, 2\}}, \quad \|D_\phi N_\varepsilon(\phi)\|_{\mathcal{L}(\mathcal{C}^*)} \le C\|\phi\|_*^{\min\{p+\varepsilon_0-1, 1\}}$$

and

$$\|\nabla_{\boldsymbol{\xi}} N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p+\varepsilon_0, 1\}}.$$

Proof. By (2.3.5), we have

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}\left((V+\phi)^{p+\varepsilon} - V^{p+\varepsilon} - (p+\varepsilon)V^{p+\varepsilon-1}\phi\right).$$

Let $f(t) = (V + t\phi)^{p+\varepsilon}$. By the Mean value Theorem, there is $\nu \in (0,1)$ such that

$$(V+\phi)^{p+\varepsilon} - V^{p+\varepsilon} = (p+\varepsilon)(V+\nu\phi)^{p+\varepsilon-1}\phi$$

and this in turn implies that

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}(p+\varepsilon)\phi\left((V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right), \tag{2.3.14}$$

for some $\nu \in (0,1)$.

Define $g(t) = (V + t\nu\phi)^{p+\varepsilon-1}$. Then, using the Mean value Theorem once again, there is $\vartheta \in (0,1)$ such that

$$(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1} = (p+\varepsilon-1)(V+\vartheta\nu\phi)^{p+\varepsilon-2}\nu\phi.$$

Thus,

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)\nu\phi^2(V+\vartheta\nu\phi)^{p+\varepsilon-2}$$

To continue, we need to study 2 cases. First, assume $p \geq 2$. Then, if $|V| \geq |\phi|$, we have

$$\begin{aligned} |N_\varepsilon(\phi)| \leq & \; C|V+\vartheta\nu\phi|^{p+\varepsilon-2}|\phi|^2 \\ \leq & \; C|\phi|^2. \end{aligned}$$

On the other hand, if $|V| \leq |\phi|$, then

$$\begin{aligned} |N_\varepsilon(\phi)| \leq & \; C|V+\vartheta\nu\phi|^{p+\varepsilon-2}|\phi|^2 \\ \leq & \; C|\phi|^{p+\varepsilon}. \end{aligned}$$

Onto the second case, when $p < 2$, we can consider $\varepsilon > 0$ sufficiently small such that $p+\varepsilon-2 < 0$. Then, from (2.3.14), we have

$$|N_\varepsilon(\phi)| \leq C|\phi| \left|(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right|.$$

Let $h(t) = (t+1)^{p+\varepsilon-1} - t^{p+\varepsilon-1}$. Differentiating, it is easy to see that

$$h'(t) = (p+\varepsilon-1)((t+1)^{p+\varepsilon-2} - t^{p+\varepsilon-2}) < 0,$$

since $p+\varepsilon-2 < 0$ and this in turn implies that $h(t)$ is decreasing. Notice that $h(0) = 1$, therefore $h(t) \leq 1$ when $t \geq 0$. Thus, if $|V+\nu\phi| > |V|$, then

$$\begin{aligned} \left|(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right| = & \; (V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1} \\ = & \; ((V+\nu\phi)-V)^{p+\varepsilon-1}\left[\left(\frac{V+\nu\phi}{(V+\nu\phi)-V}\right)^{p+\varepsilon-1}\right. \\ & \left. - \left(\frac{V}{(V+\nu\phi)-V}\right)^{p+\varepsilon-1}\right] \\ = & \; (\nu\phi)^{p+\varepsilon-1} h\left(\frac{V}{\nu\phi}\right) \\ \leq & \; |\phi|^{p+\varepsilon-1}. \end{aligned}$$

Notice that when $|V+\nu\phi| < |V|$, then

$$\left|(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right| = V^{p+\varepsilon-1} - (V+\nu\phi)^{p+\varepsilon-1}$$

and the proof is analogous to the previous case. Accordingly, $N_\varepsilon(\phi)$ can be estimate as

$$\begin{aligned} |N_\varepsilon(\phi)| \leq & \; C|\phi| \left|(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right| \\ \leq & \; C|\phi|^{p+\varepsilon-1}|\phi| \\ \leq & \; C|\phi|^{p+\varepsilon}. \end{aligned}$$

Let $\kappa(\rho) = \sup\limits_{\rho \in \mathbb{R}^+} \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}$. Thus

$$\begin{aligned} \kappa^{-1}|N_\varepsilon(\phi)| \leq&\ C\kappa^{-1}|\phi|^{\min\{p+\varepsilon,2\}} \\ \|N_\varepsilon(\phi)\|_* \leq&\ C\kappa^{\min\{p+\varepsilon,2\}-1}\kappa^{-\min\{p+\varepsilon,2\}}|\phi|^{\min\{p+\varepsilon,2\}} \\ \leq&\ C\|\phi\|_*^{\min\{p+\varepsilon,2\}}. \end{aligned}$$

For the second estimation, by direct calculus we have

$$D_\phi N_\varepsilon(\phi)\,[\psi] = e^{\varepsilon\rho}(p+\varepsilon)\left((V+\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right)\psi.$$

Following the same steps as the previous estimation, if $p \geq 2$, then

$$\begin{aligned} |D_\phi N_\varepsilon(\phi)\,[\psi]\,| =&\ e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)|V+\vartheta\nu\phi|^{p+\varepsilon-2}\nu|\phi||\psi| \\ \leq&\ C|\phi|^{\min\{p+\varepsilon-1,1\}}|\psi|. \end{aligned}$$

Conversely, if $p < 2$

$$\begin{aligned} |D_\phi N_\varepsilon(\phi)\,[\psi]\,| =&\ e^{\varepsilon\rho}(p+\varepsilon)\left|(V+\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1})\right||\psi| \\ \leq&\ C\left|(V+\nu\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1}\right||\psi| \\ \leq&\ C|\phi|^{p+\varepsilon-1}|\psi|. \end{aligned}$$

Thus, combining both cases, we conclude

$$\begin{aligned} |D_\phi N_\varepsilon(\phi)\,[\psi]\,| \leq&\ C|\phi|^{\min\{p+\varepsilon-1,1\}}|\psi| \\ \|D_\phi N_\varepsilon(\phi)\|_{\mathcal{L}(\mathcal{C}^*)} \leq&\ C\kappa^{\min\{+\varepsilon-1,1\}}\kappa^{-\min\{p+\varepsilon-1,1\}}|\phi|^{\min\{p+\varepsilon-1,1\}} \\ =&\ C\|\phi\|_*^{\min\{p+\varepsilon-1,1\}}. \end{aligned}$$

For the last estimation, by direct calculations, we have

$$\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i} = e^{p\varepsilon}(p+\varepsilon)\left((V+\phi)^{p+\varepsilon-1} - V^{p+\varepsilon-1} - (p+\varepsilon-1)V^{p+\varepsilon-2}\phi\right)\frac{\partial V}{\partial \xi_i}.$$

Using the Mean value Theorem, we obtain

$$\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| = e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)\left|(V+\nu\phi)^{p+\varepsilon-2} - V^{p+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right|.$$

If $|V| > |\phi|$, then

$$\begin{aligned} \left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| =&\ e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)\left|(V+\nu\phi)^{p+\varepsilon-2} - V^{p+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ \leq&\ C|V|^{p+\varepsilon-2}|\phi||V| \\ \leq&\ C|V|^{p+\varepsilon-1}|\phi| \\ \leq&\ C|\phi|. \end{aligned}$$

On the other hand, if $|V| < |\phi|$, we will study two cases. First, if $p \geq 2$, we can choose $\vartheta \in (0,1)$ and use the Mean Value Theorem to get

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &= e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)\left|(V+\nu\phi)^{p+\varepsilon-2} - V^{p+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ &\leq C|V+\vartheta\nu\phi|^{p+\varepsilon-3}|\phi|^2|V|.\end{aligned}$$

If $p \geq 3$, then

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &\leq C|V+\vartheta\nu\phi|^{p+\varepsilon-3}|\phi|^2|V| \\ &\leq C|\phi|^{p+\varepsilon-3}|\phi|^3 \\ &\leq C|\phi|^{p+\varepsilon}\end{aligned}$$

and if $2 \leq p < 3$

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &\leq C|V+\vartheta\nu\phi|^{p+\varepsilon-3}|\phi|^2|V| \\ &\leq C|V|^{p+\varepsilon-3}|\phi|^2|V| \\ &\leq C|V|^{p+\varepsilon-2}|\phi|^2 \\ &\leq C|\phi|^{p+\varepsilon}.\end{aligned}$$

Lastly, we need to study the case $1 < p < 2$. Notice that if we choose ε small enough such that $p+\varepsilon-2 < 0$, then

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &= e^{\varepsilon\rho}(p+\varepsilon)(p+\varepsilon-1)\left|(V+\nu\phi)^{p+\varepsilon-2} - V^{p+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ &\leq C|V|^{p+\varepsilon-2}|\phi||V| \\ &\leq C|V|^{p+\varepsilon-1}|\phi| \\ &\leq C|\phi|^{p+\varepsilon}\end{aligned}$$

and this allows us to conclude that, in both cases, the following estimate holds

$$\begin{aligned}\kappa^{-1}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &\leq C\kappa^{-1}|\phi|^{\min\{p+\varepsilon,1\}} \\ \|\nabla_{\boldsymbol{\xi}} N_\varepsilon(\phi)\|_* &\leq C\kappa^{\min\{p+\varepsilon,1\}-1}\kappa^{-\min\{p+\varepsilon,1\}}|\phi|^{\min\{p+\varepsilon,1\}} \\ &\leq C\|\phi\|_*^{\min\{p+\varepsilon,1\}}.\end{aligned}$$

□

In the same vein, we can also prove the estimations for R_ε.

Lemma 2.3.5. *Let $\xi_1, \xi_2, \ldots, \xi_k$ satisfying (2.3.13). Assume additionally that σ is sufficiently small. Then*

$$\|R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}} \text{ and } \|\nabla_{\boldsymbol{\xi}} R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}},$$

for some $\overline{\sigma} \in (0, \frac{1}{2})$.

Proof. From (2.3.6), we can rewrite R_ε as

$$\begin{aligned} R_\varepsilon = & \ (e^{\varepsilon\rho}V^{p+\varepsilon} - e^{\varepsilon\rho}V^p) + (e^{\varepsilon\rho}V^p - V^p) + (V^p - \sum_{i=1}^{k} W_i^p) + \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho}V \\ = & \ A_1 + A_2 + A_3 + A_4. \end{aligned}$$

For $A_1(\rho)$, define $f(t) = V^{p+t}$. Thus, by Mean Value Theorem, we have

$$A_1(\rho) = -\varepsilon e^{\varepsilon\rho}\log(V(\rho))V^{p+\nu},$$

for some $\nu \in (-\varepsilon, 0)$.
If $|V(\rho)| \geq 1$, then

$$|A_1(\rho)| \leq C\varepsilon|V(\rho)|^{p+1}$$

and this in turns implies that $\|A_1\|_* \leq C\varepsilon\|V\|_* \leq C\varepsilon$.
On the other hand, if $|V(\rho)| \leq 1$, we can define $s(t) = t^{p+\nu-1}\log(t)$, with $t \in [0,1]$. Note that this function reaches its maximum at $t = e^{-\frac{1}{p+\nu-1}}$, so upon replacement, we obtain

$$\begin{aligned} |A_1(\rho)| \leq & \ C\varepsilon|\log(V(\rho))|\,|V^{p+\nu}| \\ \leq & \ C\varepsilon|\log(V(\rho))|\,|V^{p+\nu-1}|\,|V(\rho)| \\ \leq & \ C\varepsilon\tfrac{1}{e(p+\nu-1)}|V(\rho)| \\ \leq & \ C\varepsilon|V(\rho)| \end{aligned}$$

and therefore $\|A_1\|_* \leq C\varepsilon\|V\|_* \leq C\varepsilon$.

For $A_2(\rho)$, define $g(t) = e^{t\rho}$. By Mean Value Theorem,

$$A_2(\rho) = -\varepsilon\rho e^{p\nu}V^p,$$

for some $\nu \in (-\varepsilon, 0)$. To continue, we need to analyze two cases. First, if $\rho \leq \xi_k$, then

$$\begin{aligned} \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|} \geq & \ e^{-\sigma|\rho-\xi_k|} \\ = & \ e^{-\sigma(\xi_k-\rho)} \end{aligned}$$

and such

$$\left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} \leq e^{\sigma(\xi_k-\rho)}.$$

Even more, from (2.3.13), we have $e^{\sigma\xi_k} \leq (M\varepsilon)^{-k\sigma}$. Thus

$$\begin{aligned}
\left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} |A_2(\rho)| \leq & \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} \varepsilon\rho|V|^p \\
\leq & \ e^{\sigma(\xi_k-\rho)}\varepsilon\rho|V|^p \\
\leq & \ \varepsilon\|V^p\|_\infty(\rho e^{-\sigma\rho})e^{\sigma\xi_k} \\
\leq & \ C\varepsilon(M\varepsilon)^{-k\sigma} \\
\leq & \ C\varepsilon^{1-k\sigma}.
\end{aligned}$$

Conversely, if $\rho > \xi_k$, then

$$\begin{aligned}
\frac{|V(\rho)|}{\sum\limits_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}} \leq & \ \frac{Ce^{-|\rho-\xi_k|}}{ke^{-\sigma|\rho-\xi_k|}} \\
= & \ Ce^{-(1-\sigma)(\rho-\xi_k)}
\end{aligned}$$

and such

$$\begin{aligned}
\left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} |A_2(\rho)| \leq & \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} \varepsilon\rho|V|^p \\
\leq & \ C\varepsilon\rho e^{-(p-\sigma)(\rho-\xi_k)} \\
\leq & \ C\varepsilon\rho e^{-\sigma(\rho-\xi_k)} \\
= & \ C\varepsilon(\rho e^{-\sigma\rho})e^{\sigma\xi_k} \\
\leq & \ C\varepsilon^{1-k\sigma},
\end{aligned}$$

for σ small enough. Therefore, $\|A_2\|_* \leq C\varepsilon^{1-k\sigma}$. Note that if we choose $\sigma \in (\frac{1}{2k}, \frac{1}{k})$, then $1 - k\sigma \in (0, \frac{1}{2})$.

Next, to estimate $A_3(\rho)$, we need to introduce a positive constant $\omega \in (0, 1)$, which will be determine later. Assume that there is $l \in \{1, \ldots, k\}$ such that $|\rho - \xi_l| < \omega \log(\frac{1}{M\varepsilon})$. From (2.3.13), we have

$$\begin{aligned}
|\rho - \xi_i| > & \ |\xi_l - \xi_i| - |\rho - \xi_l| \\
> & \ \log(\tfrac{1}{M\varepsilon}) - \omega\log(\tfrac{1}{M\varepsilon}) \\
= & \ (1-\omega)\log(\tfrac{1}{M\varepsilon})
\end{aligned}$$

and this implies that $e^{-|\rho-\xi_i|} \leq M\varepsilon^{(1-\omega)}$, for $i \neq l$. Even more, the previous inequality implies that $|W_i| \leq M\varepsilon^{(1-\omega)}, |\Pi_i| \leq M\varepsilon^{(1-\omega)}$ and $|V_i| \leq M\varepsilon^{(1-\omega)}$ if $i \neq l$. Moreover, we notice that $|\rho - \xi_l| < \omega\log(\frac{1}{M\varepsilon})$ also indicates

$$\begin{aligned}
-\omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho - \xi_l \\
2\xi_l - \omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho + \xi_l \\
2\xi_1 - \omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho + \xi_l,
\end{aligned}$$

thus

$$\begin{aligned}|\Pi_l(\rho)| \leq\ & Ce^{-(\rho+\xi_l)}\\ \leq\ & Ce^{-2\xi_1+\omega\log(\frac{1}{M\varepsilon})}\\ \leq\ & Ce^{-2\xi_1}\varepsilon^{-\omega}\\ \leq\ & C\varepsilon^{1-\omega}.\end{aligned}$$

Having stated that, define

$$h(t) = \left(W_l(\rho) + t\left(\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho)\right)\right)^p,$$

then, by Mean Value Theorem, we have

$$V^p(\rho) - W_l(\rho) = p\left(W_l(\rho) + \nu\left(\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho)\right)\right)^{p-1}\left(\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho)\right),$$

for some $\nu \in (0,1)$. Hence

$$\begin{aligned}\|A_3(\rho)\|_* &= \kappa^{-1}\left| p\left(W_l(\rho)+\nu\left(\Pi_l(\rho)+\sum_{i=1, i\neq l}^{k} V_i(\rho)\right)\right)^{p-1}\left(\Pi_l(\rho)+\sum_{i=1, i\neq l}^{k} V_i(\rho)\right) - \sum_{i=1, i\neq l}^{k} W_i^p(\rho)\right|\\ &\leq\ C\left|\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho)\right| + \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1}\left|\sum_{i=1, i\neq l}^{k} W_i^p(\rho)\right|\\ &\leq\ C\varepsilon^{(1-\omega)} + C\sum_{i=1, i\neq l}^{k} e^{-(p-\sigma)|\rho-\xi_i|}\\ &\leq\ C\varepsilon^{(1-\omega)} + C\varepsilon^{(p-\sigma)(1-\omega)}\\ &\leq\ C\varepsilon^{(1-\omega)}.\end{aligned}$$

Now assume the opposite, in other words, let $|\rho - \xi_i| \geq \omega\log(\frac{1}{M\varepsilon})$ for all $i \in \{1, \dots, k\}$. It is easy to see that this implies $e^{-|\rho-\xi_i|} \leq M\varepsilon^{\omega}$ for all $i \in \{1, \dots, k\}$. This in turn implies that

$$\begin{aligned}V_i(\rho) \leq\ & Ce^{-|\rho-\xi_i|}\\ \leq\ & C\varepsilon^{\omega},\end{aligned}$$

for all $i \in \{1, \dots, k\}$ and the same applies to $W_i(\rho)$ for all $i \in \{1, \dots, k\}$. Thus, we can estimate

$$\begin{aligned}\|A_3\|_* \leq\ & \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1}|V(\rho)|^p + \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1}\sum_{i=1}^{k}|W_i(\rho)|^p\\ \leq\ & C\varepsilon^{-\sigma\omega}\varepsilon^{p\omega} + C\varepsilon^{-\sigma\omega}\varepsilon^{p\omega}\\ =\ & C\varepsilon^{(p-\sigma)\omega}.\end{aligned}$$

Choosing ω such that $(p-\sigma)\omega = 1-\omega$, in other words $\omega = \frac{1}{p-\sigma+1}$, we can conclude that $\|A_3\|_* \leq$

$C\varepsilon^{(1-\omega)}$. Note that if we choose $\sigma < p-1$, then $\omega \in (0, \frac{1}{2})$ and this in turn implies that $1-\omega \in (\frac{1}{2}, 1)$.

Lastly, for $A_4(\rho)$, we have

$$\begin{aligned}\left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} |A_4(\rho)| \le{}& \eta\varepsilon^{\frac{N-8}{N-4}} a_\varepsilon e^{-\frac{8}{N-4}\rho} \sum_{i=1}^{k} e^{-|\rho-\xi_i|} \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} \\ \le{}& \eta\varepsilon^{\frac{N-8}{N-4}} a_\varepsilon e^{-\frac{8}{N-4}\rho} \sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|} \left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|}\right)^{-1} \\ \le{}& C\varepsilon^{\frac{N-8}{N-4}},\end{aligned}$$

thus $\|A_4\|_* \le C\varepsilon^{\frac{N-8}{N-4}}$. Moreover, if $N < 12$, then $\frac{N-8}{N-4} \in (0, \frac{1}{2})$. Therefore, $\|R_\varepsilon\|_* \le C\varepsilon^{\overline{\sigma}}$, where $\overline{\sigma} = \min\{(1-k\sigma), \frac{N-8}{N-4}\}$.

To prove $\|\nabla_{\boldsymbol{\xi}} R_\varepsilon\|_* \le C\varepsilon^{\overline{\sigma}}$, notice that

$$\frac{\partial R_\varepsilon}{\partial \xi_i} = \frac{\partial A_1}{\partial \xi_i} + \frac{\partial A_2}{\partial \xi_i} + \frac{\partial A_3}{\partial \xi_i} + \frac{\partial A_4}{\partial \xi_i},$$

for all $i = \{1, \ldots, k\}$. Then, we can use the previous procedure to conclude the proof. □

With the two previous lemmas, we are able to achieve the goal of this section. To do this, we state the following proposition

Proposition 2.3.6. *Assume that the constraints (2.3.13) are satisfied. Then, there exists $C > 0$ such that for every ε small enough, there exists an unique solution $\phi = \phi(\boldsymbol{\xi})$ to the problem (2.3.8). Furthermore, the map $\boldsymbol{\xi} \mapsto \phi(\boldsymbol{\xi})$ is of class C^1 with respect to the norm $\|\cdot\|_*$ and satisfies*

$$\|\phi\|_* \le C\varepsilon^{\overline{\sigma}}, \quad \|\nabla_{\boldsymbol{\xi}}\phi\|_* \le C\varepsilon^{\overline{\sigma}},$$

for some $\overline{\sigma} \in \left(0, \frac{1}{2}\right)$.

Proof. Let the operator $F_\varepsilon : \mathcal{A}_r \to \mathcal{C}^*(\mathcal{Q})$ be defined as

$$F_\varepsilon(\phi) = T_\varepsilon(N_\varepsilon(\phi) + R_\varepsilon),$$

where T_ε is given by Proposition 2.3.2, and

$$\mathcal{A}_r = \{\phi \in \mathcal{C}^*((0,\infty)) : \|\phi\|_* \le r\varepsilon^{\overline{\sigma}}\},$$

for a suitable $r > 0$ to be chosen later. Note that if it is possible to show that F_ε is a contraction, then it follows that there is a fixed point in $\mathcal{A}_r$ for F_ε, which is equivalent to solving the problem (2.3.8). From Proposition 2.3.2, Proposition 2.3.3, Lemma 2.3.4 and Lemma 2.3.5, we obtain

$$\begin{aligned}\|F_\varepsilon(\phi)\|_* \le{}& C\,\|N_\varepsilon(\phi) + R_\varepsilon\|_* \\ \le{}& C\left((r\varepsilon^{\overline{\sigma}})^{\min\{p+\varepsilon,2\}} + r\varepsilon^{\overline{\sigma}}\right) \\ \le{}& r\varepsilon^{\overline{\sigma}},\end{aligned}$$

for some appropriately chosen $r > 0$. Noticing that

$$\|F_\varepsilon(\phi_1) - F_\varepsilon(\phi_2)\|_* \leq C \|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_*,$$

for $\phi_1, \phi_2 \in \mathcal{A}_r$, it can be asserted that F_ε is a contraction if N_ε is. Indeed, by the Mean value Theorem and Lemma 2.3.4,

$$\|N_\varepsilon(\phi_1) - N_\varepsilon(\phi_2)\|_* \leq C\varepsilon^{\overline{\sigma}\min\{p+\varepsilon-1,1\}} \|\phi_1 - \phi_2\|_*,$$

and this in turn implies that

$$\|F_\varepsilon(\phi_1) - F_\varepsilon(\phi_2)\|_* \leq C(r\varepsilon^{\overline{\sigma}})^{\min\{p+\varepsilon-1,1\}} \|\phi_1 - \phi_2\|_*,$$

thus F_ε is a contraction. By Banach fixed point Theorem, there is $\phi \in \mathcal{A}_r$ such that ϕ is a solution of problem (2.3.8).
As for its differentiability, direct calculations yield

$$\nabla_{\boldsymbol{\xi}}\phi(\boldsymbol{\xi}) = \nabla_{\boldsymbol{\xi}}T_\varepsilon(N_\varepsilon(\phi(\boldsymbol{\xi})) + R_\varepsilon(\boldsymbol{\xi})) + T_\varepsilon(\nabla_{\boldsymbol{\xi}}N_\varepsilon(\phi(\boldsymbol{\xi})) + \nabla_{\boldsymbol{\xi}}R_\varepsilon(\boldsymbol{\xi})),$$

thus, by Proposition 2.3.2, Proposition 2.3.3, Lemma 2.3.4 and Lemma 2.3.5

$$\begin{aligned}\|\nabla_{\boldsymbol{\xi}}\phi(\boldsymbol{\xi})\|_* \leq&\ C\|N_\varepsilon(\phi(\boldsymbol{\xi})) + R_\varepsilon(\boldsymbol{\xi})\|_* + \|\nabla_{\boldsymbol{\xi}}N_\varepsilon(\phi(\boldsymbol{\xi})) + \nabla_{\boldsymbol{\xi}}R_\varepsilon(\boldsymbol{\xi})\| \\ \leq&\ C\varepsilon^{\overline{\sigma}}.\end{aligned}$$

□

2.4 Finite dimensional reduction

In this section, we will consider the constraints (2.3.13) and the function $\phi = \phi(\boldsymbol{\xi})$ given by Proposition 2.3.6, which is the unique solution to the problem (2.3.8). According to the previous section, we know that $c_i = 0$ in (2.3.8) for each $i = 1, \ldots, k$ is equivalent to stating that $v_* = V + \phi(\boldsymbol{\xi})$ is a solution to the problem (2.1.11). Therefore, it is necessary to find $\boldsymbol{\xi} \in \mathbb{R}^k$ such that the system $c_i(\boldsymbol{\xi}) = 0$ for every $i = 1, \ldots, k$ has a solution. For this, define the functional

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) := E_\varepsilon(V + \phi(\boldsymbol{\xi})), \tag{2.4.1}$$

where E_ε is the functional defined in (2.1.13). If it was possible to find $\boldsymbol{\xi}$ such that it is a critical point of $\mathcal{F}_\varepsilon(\boldsymbol{\xi})$, then we would be able to assert that $v_* = V + \phi(\boldsymbol{\xi})$ is a solution to the problem (2.1.11). To organize the relationships written above, the following lemma is stated

Lemma 2.4.1. *Let* $\boldsymbol{\xi} = (\xi_1, \xi_2, \ldots, \xi_k)$ *satisfying conditions (2.3.13). Then,* $\boldsymbol{\xi}$ *is a critical point of* $\mathcal{F}_\varepsilon$ *if and only if* $c_i(\boldsymbol{\xi}) = 0$ *for every* $i = 1, \ldots, k$.

Proof. Assume that $\boldsymbol{\xi_i}$ satisfies (2.3.13) and $c_i(\boldsymbol{\xi}) = 0$ for every $i = 1, \ldots, k$. Then, $v_* = V + \phi(\boldsymbol{\xi})$ is a solution to the problem (2.1.11) and this in turn implies that

$$\nabla_{\boldsymbol{\xi}}\mathcal{F}_\varepsilon(V + \phi(\boldsymbol{\xi})) = 0,$$

then, by (2.4.1)

$$\frac{\partial \mathcal{F}_\varepsilon(\boldsymbol{\xi})}{\partial \xi_i} = DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\left[\frac{\partial}{\partial \xi_i}(V+\phi(\boldsymbol{\xi}))\right],$$

for all $i=1,\dots,k$. Accordingly, $\boldsymbol{\xi}$ is a critical point of $\mathcal{F}_\varepsilon$.
Conversely, assume $\boldsymbol{\xi}$ is a critical point of $\mathcal{F}_\varepsilon$. Choose $l \in \{1,\dots,k\}$ arbitrary but fixed, then

$$\frac{\partial \mathcal{F}_\varepsilon(\boldsymbol{\xi})}{\partial \xi_l} = 0$$

and thus

$$DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\left[\frac{\partial}{\partial \xi_l}(V+\phi(\boldsymbol{\xi}))\right] = DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\left[Z_l + \frac{\partial \phi(\boldsymbol{\xi})}{\partial \xi_l}\right] = 0.$$

From (2.3.6), we have $\|\frac{\partial \phi}{\partial \xi_l}\|_* \to 0$ as $\varepsilon \to 0$. Then, there exist constants $a_1^l, \dots, a_k^l$ such that

$$\int_0^\infty \frac{\partial \phi}{\partial \xi_l} Z_i \, d\rho = \sum_{j=1}^k a_j^l \int_0^\infty Z_i Z_j \, d\rho.$$

Given that $\|\frac{\partial \phi}{\partial \xi_l}\|_* \to 0$, then $a_j^l \to 0$ for $j=1,\dots,k$. So, we can rewrite $\frac{\partial \phi}{\partial \xi_l}$ as

$$\frac{\partial \phi}{\partial \xi_l} = \sum_{j=1}^k a_j^l Z_j + \left(\frac{\partial \phi}{\partial \xi_l} - \sum_{j=1}^k a_j^l Z_j\right) = p + q,$$

where $p \in \operatorname{span}\{Z_1,\dots,Z_k\}$ and

$$\int_0^\infty q Z_i \, d\rho = 0.$$

This implies that $DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\,[q] = 0$. Henceforth

$$DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\,[Z_l + p] = DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\left[(1+a_l^l)Z_l + \sum_{j=1, j\neq l}^k a_j^l Z_l\right] = 0.$$

Note that the choice of l was arbitrarily selected, so the above equality holds for every $l=1,\dots,k$. This allows us to study the associated linear system, which is diagonally dominant, and therefore we can conclude that

$$DE_\varepsilon(V+\phi(\boldsymbol{\xi}))\,[Z_l] = 0,$$

for every $l=1,\dots,k$. Thus, upon integrating, we arrive at

$$\int_0^\infty \Big((V+\phi(\boldsymbol{\xi}))^{(4)} - C_4(V+\phi(\boldsymbol{\xi}))'' + C_5(V+\phi(\boldsymbol{\xi})) - e^{\varepsilon\rho}(V+\phi(\boldsymbol{\xi}))^{p+\varepsilon} - \lambda_\varepsilon a_\varepsilon e^{-\frac{8}{N-4}\rho}(V+\phi(\boldsymbol{\xi}))\Big) Z_l \, d\rho = 0,$$

for every $l = 1, \ldots, k$. But $\phi(\boldsymbol{\xi})$ is the solution of (2.3.3), thus

$$\sum_{i=1}^{k} c_i \int_0^\infty Z_i Z_j \, d\rho = 0,$$

for every $j = 1, \ldots, k$ and this in turn implies that $c_i(\boldsymbol{\xi}) = 0$ for every $i = 1, \ldots, k$. □

The final step in this section is to validate an expansion for $\mathcal{F}_\varepsilon$, which will be crucial for finding its critical points.

Lemma 2.4.2. *Under the assumptions of 2.3.13 and considering $\phi = \phi(\boldsymbol{\xi})$ as in Proposition 2.3.6, the following expansion holds*

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V) + o(\varepsilon),$$

where $o(\varepsilon)$ is uniformly small in the C^1 sense in the vectors $\boldsymbol{\xi}$.

Proof. Notice that

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V) = E_\varepsilon(V + \phi) - E_\varepsilon(V).$$

After applying the Fundamental Theorem of Calculus and integrating by parts, it follows that

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V) = \int_0^1 \nabla E_\varepsilon(V + t\phi)\,[\phi]\, dt = -\int_0^1 t D^2 E_\varepsilon(V + t\phi)\,[\phi]\,[\phi]\, dt.$$

From (2.1.13), we have

$$\begin{aligned}
D^2 E_\varepsilon(V+t\phi)[\phi][\phi] = & \int_0^\infty (\phi''(\rho)^2 + C_4 \phi'(\rho)^2 + C_5 \phi(\rho)^2)\, d\rho + \frac{4}{N-4}\phi'(0)^2 \\
& - (p+\varepsilon)\int_0^\infty e^{\varepsilon\rho}(V+t\phi)^{p+\varepsilon-1}(\rho)\, d\rho - \lambda_\varepsilon a_\varepsilon \int_0^\infty e^{-\frac{8}{N-4}\rho}\phi(\rho)^2\, d\rho \\
= & \int_0^\infty (\phi^{(4)}(\rho) + C_4\phi''(\rho) + C_5\phi(\rho))\phi(\rho)\, d\rho \\
& - (p+\varepsilon)\int_0^\infty e^{\varepsilon\rho}(V+t\phi)^{p+\varepsilon-1}(\rho)\, d\rho - \lambda_\varepsilon a_\varepsilon \int_0^\infty e^{-\frac{8}{N-4}\rho}\phi(\rho)^2\, d\rho \\
= & \int_0^\infty L_\varepsilon(\phi)\phi\, d\rho + (p+\varepsilon)\int_0^\infty e^{\varepsilon\rho}\left(V^{p+\varepsilon-1} - (V+t\phi)^{p+\varepsilon-1}\right)\phi^2(\rho)\, d\rho
\end{aligned}$$

and such, by (2.3.3), (2.3.4), (2.3.5) and (2.4.1)

$$\begin{aligned}
|\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V)| = & \left|\int_0^1 t\int_0^\infty L_\varepsilon(\phi)\phi\, d\rho + (p+\varepsilon)\int_0^1 t\int_0^\infty e^{\varepsilon\rho}\left(V^{p+\varepsilon-1} - (V+t\phi)^{p+\varepsilon-1}\right)\phi^2(\rho)\, d\rho\right| \\
\leq & \; C\left(\|N_\varepsilon\|_* + \|R_\varepsilon\|_*\right)\|\phi\|_* + C\|\phi\|_*^2 \\
\leq & \; C(\varepsilon^{\overline{\sigma}} + \varepsilon^{\overline{\sigma}})\varepsilon^{\overline{\sigma}} + C\varepsilon^{\overline{2\sigma}} \\
\leq & \; C\varepsilon^{2\overline{\sigma}},
\end{aligned}$$

but $\overline{\sigma} \in (0, \frac{1}{2})$, thus

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - E_\varepsilon(V) = o(\varepsilon).$$

Regarding differentiability, through direct calculations, it can be shown that

$$\begin{aligned}\frac{\partial}{\partial \xi_i}(\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - I_\varepsilon(V)) = & -\int_0^1 t\left(\int_0^\infty \frac{\partial}{\partial \xi_i}((N_\varepsilon(\phi) + R_\varepsilon)\phi)\, d\rho\right) dt \\ & -(p+\varepsilon)\int_0^1 t\left(\int_0^\infty e^{\varepsilon\rho}\frac{\partial}{\partial \xi_i}\left(V^{p+\varepsilon-1} - (V+t\phi)^{p+\varepsilon-1}\right)\phi^2(\rho)\, d\rho\right) dt,\end{aligned}$$

then, using once again (2.3.4) and (2.3.5), we have

$$\frac{\partial}{\partial \xi_i}\left(\mathcal{F}_\varepsilon(\boldsymbol{\xi}) - I_\varepsilon(V)\right) = o(\varepsilon)$$

and this ends the proof. □

2.5 Proof of Theorem 1.4.1

Proof of Theorem 1.4.1. From the definition of $\mathcal{F}\varepsilon$ in (2.4.1) and Lemma 2.4.1, it is clear that to conclude the proof, it will be sufficient to find a critical point for the functional $\Phi_\varepsilon : \mathbb{R}^k \to \mathbb{R}$ defined by

$$\Phi_\varepsilon(\boldsymbol{\mu}) := \varepsilon^{-1}\mathcal{F}_\varepsilon(\boldsymbol{\xi}(\boldsymbol{\mu})). \tag{2.5.1}$$

From Lemma (2.2.2) and (2.4.2), it is observed that

$$\nabla\Phi_\varepsilon(\boldsymbol{\mu}) = \nabla\Psi_k(\boldsymbol{\mu}) + \vec{o}(1),$$

where $\vec{o}(1) \to 0$ uniformly as $\varepsilon \to 0$ and the functional Ψ_k is the one defined in (2.2.4).

It is now observed that Ψ_k has two non-degenerate critical points $\boldsymbol{\mu}^\pm = \left(\mu_1^\pm, \mu_2^*, \ldots, \mu_k^*\right)$, where $\mu_1^\pm, \mu_i^*$ are, respectively, the critical points in $]0, +\infty[$ of the functions

$$\mu_1 \mapsto a_5\frac{1}{\mu_1^2} - a_2 k\log\mu_1 - a_6\eta\mu_1^{-\frac{8}{N-4}}$$

and

$$\mu_i \mapsto a_2(k-i+1)\log\mu_i - a_3\mu_i \;\text{ for } i = 2, 3, \ldots, k,$$

where a_js are given by (2.2.2).

Indeed, let

$$f(\mu_1) = a_5\frac{1}{\mu_1^2} - a_2 k\log\mu_1 - a_6\eta\mu_1^{-\frac{8}{N-4}}.$$

Differentiating f and equating to zero, we obtain

$$-2a_5\mu_1^{-3} - ka_2\mu_1^{-1} + \frac{8}{N-4}a_6\eta\mu_1^{-\frac{8}{N-4}-1} = 0.$$

Then,

$$\eta = \frac{a_5}{a_6}\frac{N-4}{4}\mu_1^{\frac{8}{N-4}-2} + \frac{a_2}{a_6}\frac{k(N-4)}{8}\mu_1^{\frac{8}{N-4}}.$$

Now, define

$$g(s) = \frac{a_5}{a_6}\frac{N-4}{4}s^{\frac{8}{N-4}-2} + \frac{a_2}{a_6}\frac{k(N-4)}{8}s^{\frac{8}{N-4}}$$

and let $\overline{\eta}_k$ be the minimum of $g(s)$, which is attained at

$$s = \left(\frac{a_5}{a_2}\frac{1}{k}\left(\frac{(N-4)}{2} - 2\right)\right)^{1/2}.$$

Then, if $\eta > \overline{\eta}_k$, $f(\mu_1)$ has a strict maximum point $\mu_1^+ = \mu_1^+(\eta)$, and a strict minimum point $\mu_1^- = \mu_1^-(\eta)$. To check this, notice that

$$\frac{d^2 f}{d\mu_1}(\mu_1^+) < 0 \quad \text{y} \quad \frac{d^2 f}{d\mu_1}(\mu_1^-) > 0.$$

On the other hand, the function

$$h_i(\mu_i) = a_2(k-i+1)\log\mu_i - a_3\mu_i$$

has a unique strict maximum point, μ_i^* given by

$$\mu_i^* = \frac{a_2}{a_3}(k-i+1) \quad \text{for } i = 2, 3, \ldots, k.$$

Indeed,

$$\frac{d^2 h_i}{d\mu_i}(\mu_i^*) < 0 \quad \text{for } i = 2, 3, \ldots, k.$$

Therefore

$$\boldsymbol{\mu}^{\pm} = \left(\mu_1^{\pm}, \frac{a_2}{a_3}(k-1), \frac{a_2}{a_3}(k-2), \ldots, \frac{a_2}{a_3}\right),$$

are the only critical points of Ψ_k, which are non degenerate.

Following the above, we have that $\nabla\Psi_k$ is stable with respect to small and uniform perturbations. Consequently, if $\mathfrak{V}_{\pm}$ are small and arbitrary neighborhoods of points $\boldsymbol{\mu}^{\pm}$ in $\mathbb{R}^k$, then the topological degrees $\deg(\nabla\Psi_k, \mathfrak{V}_{\pm}, 0)$ are well defined and

$$\deg(\nabla\Psi_k, \mathfrak{V}_{\pm}, 0) \neq 0.$$

Even more, since for $\varepsilon > 0$ the expansion

$$\nabla\Phi_\varepsilon(\boldsymbol{\mu}) = \nabla\Psi_k(\boldsymbol{\mu}) + \vec{o}(1),$$

is true, with $\vec{o}(1) \to \vec{0}$ uniformly in $\mathfrak{V}_{\pm}$, we can use the homotopy

$$\mathcal{H}_t = t\nabla\Phi_\varepsilon + (1-t)\nabla\Psi_k,$$

defined in $\mathfrak{V}_{\pm}$ for $0 \leq t \leq 1$, to obtain

$$\deg(\nabla\Phi_\varepsilon, \mathfrak{V}_{\pm}, 0) = \deg(\nabla\Psi_k, \mathfrak{V}_{\pm}, 0) \neq 0,$$

for all small enough $\varepsilon > 0$.

Accordingly, for all small enough $\varepsilon > 0$, there exist two critical points $\boldsymbol{\mu}_\varepsilon^\pm = \left(\mu_{1,\varepsilon}^\pm, \mu_{2,\varepsilon}^\pm, \dots, \mu_{k,\varepsilon}^\pm\right)$ of Φ_ε satysfing

$$\mu_{1,\varepsilon}^\pm = \mu_1^\pm + o(1) \quad \text{and} \quad \mu_{i,\varepsilon}^\pm = \mu_i^* + o(1) \quad \text{for } i = 2, 3, \dots, k,$$

where $o(1) \to 0$ if $\varepsilon \to 0$. Using the change of variable (2.2.2), this is equivalent to saying $\boldsymbol{\xi}_\varepsilon^\pm = \left(\xi_{1,\varepsilon}^\pm, \xi_{2,\varepsilon}^\pm, \dots, \xi_{k,\varepsilon}^\pm\right)$, where

$$\xi_{1,\varepsilon}^\pm = \log \frac{\mu_{1,\varepsilon}^\pm}{\sqrt{\varepsilon}} \qquad \text{and} \qquad \xi_{i,\varepsilon}^\pm = \log \frac{\mu_{1,\varepsilon}^\pm}{\mu_{2,\varepsilon}^\pm \cdots \mu_{i,\varepsilon}^\pm \varepsilon^{\frac{2i-1}{2}}} \quad \text{for } i = 2, 3, \dots, k,$$

are the critical points of Φ_ε and this in turns implies that they are also critical points of

$$\mathcal{F}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V + \phi(\boldsymbol{\xi})).$$

Therefore, the functions

$$v_* = v_{k,\varepsilon}^\pm(\rho) = \sum_{i=1}^k \left(W(\rho - \xi_{i,\varepsilon}^\pm) + \Pi_{\xi_{i,\varepsilon}^\pm}(\rho)\right) + \phi(\boldsymbol{\xi}_\varepsilon^\pm)(\rho),$$

are solutions of (2.1.11). Then, going back to the original variables, we can conclude that

$$u_{k,\varepsilon}^\pm(x) = \overline{\gamma}_{N,2} \sum_{i=1}^k \left(\frac{e^{\frac{2}{N-4}\xi_{i,\varepsilon}^\pm}}{e^{\frac{4}{N-4}\xi_{i,\varepsilon}^\pm} + |x|^2} \right)^{\frac{N-4}{2}} + \pi_{\mu_{i,\varepsilon}^\pm}(x) + \varphi_\varepsilon(x),$$

for all $x \in B_1$, where $\mu_{i,\varepsilon}^\pm = e^{\frac{2}{N-4}\xi_{i,\varepsilon}^\pm} = M_i \varepsilon^{\frac{1-2i}{N-4}}$ and $\varphi_\varepsilon(x) \to 0$ uniformly in B_1 when $\varepsilon \to 0$, are solutions of (2.1.1). This ends the proof of Theorem 1.4.1. $\square$

Chapter 3

Positive solutions for some almost critical Brezis-Nirenberg type problems in bounded and exterior domains

3.1 Ansatz

Our intention is to find a suitable ansatz for a function u that solves the problem

$$\begin{cases} -\Delta u = |x|^{\alpha} u^{p_\alpha^* + \varepsilon} + \lambda_\varepsilon |x|^{\beta} u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1.1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $N \geq 4$, which is symmetric with respect to $x_1, \ldots, x_N$ and contains the origin, $\alpha > -2$, $-2 < \beta < N - 4$, $p_\alpha^* = \frac{N+2\alpha+2}{N-2}$, $\varepsilon > 0$ is a small parameter and $\lambda_\varepsilon > 0$ depends of ε such that $\lambda_\varepsilon \to 0$ as ε goes to 0.

In contrast to the result found by Liu [27], the influence of the regular part of the Green function associated to Ω under our assumptions cannot be avoided, so it is suitable to introduce the projection onto $H_0^1(\Omega)$ of the function $U_{\mu,\alpha}$, defined in (1.2.11), that here we denote by $P_\Omega U_{\mu,\alpha}$, the unique solution of the problem

$$\begin{cases} -\Delta P_\Omega U_{\mu,\alpha} = |x|^{\alpha} U_{\mu,\alpha}^{p_\alpha^*} & \text{in } \Omega, \\ P_\Omega U_{\mu,\alpha} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.2}$$

Moreover, as was observed in [22], when α is an even integer, the study of the limit problem (1.2.11) seems to require additional ideas. In this way, as in [21], if $\alpha = 2(m-1)$ for some $m \in \mathbb{N}$, $m \geq 2$, we consider the value

$$\sigma_m := \max\{\hat{\sigma}_k(Y) : Y \in \mathbb{Y}_m(\mathbb{R}^N)\},$$

where $\mathbb{Y}_m(\mathbb{R}^N)$ is the space of all homogeneous harmonic polynomials of degree m in $\mathbb{R}^N$ and

$$\hat{\sigma}_m(Y) := \max\Big\{h \in \mathbb{N} : Y(r,\theta) = Y\Big(r, \theta + \frac{2\pi}{h}\Big)\Big\}.$$

With those two ingredients, we are ready to find a suitable ansatz for (3.1.1).

Remark 3.1.1. *Due to the similarities in the steps required to prove Theorem 1.4.1 and Theorem 1.4.2, some proofs in this chapter will be omitted.*

3.1.1 Preliminaries

To establish an appropriate ansatz for solutions of (3.1.1), we start by considering the unique function $\pi_{\mu,\alpha} : \Omega \to \mathbb{R}$ that solves the problem

$$\begin{cases} -\Delta \pi_{\mu,\alpha} = 0 & \text{in } \Omega, \\ \pi_{\mu,\alpha} = -U_{\mu,\alpha} & \text{on } \partial\Omega, \end{cases}$$

where $U_{\mu,\alpha}$ is the bubble of order α given by (1.2.12). Hence, $P_\Omega U_{\mu,\alpha} = U_{\mu,\alpha} + \pi_{\mu,\alpha}$ solves (3.1.2) and, from [21, Proposition 3.1], we get that $0 < P_\Omega U_{\mu,\alpha} \le U_{\mu,\alpha}$, $-U_{\mu,\alpha} \le \pi_{\mu,\alpha} \le 0$ and

$$\pi_{\mu,\alpha}(y) = -\mu^{\frac{N-2}{2}} H(0,y) + O(\mu^{\frac{N+2+2\alpha}{2}}) \quad \text{for all } y \in \Omega,$$

where H is the regular part of the Green's function of the Laplacian on Ω under Dirichlet zero boundary condition, *i.e.*, for every $y \in \Omega$, $H(\cdot, y)$ solves the problem,

$$\begin{cases} -\Delta H(\cdot, y) = 0 & \text{in } \Omega, \\ H(\cdot, y) = |\cdot - y|^{N-2} & \text{on } \partial\Omega. \end{cases}$$

Thus, we seek solutions $u_{k,\varepsilon} : \Omega \to \mathbb{R}$ to (3.1.1) of the form (1.4.4), where $\varphi_\varepsilon : \Omega \to \mathbb{R}$ is a function of small order in Ω for suitable positive parameters μ_i for $i = 1, 2, \dots, k$. Note that $u_{k,\varepsilon}$ becomes a solution to (3.1.1) if it is a stationary point of the associated energy functional J_ε defined by

$$\mathcal{E}_\varepsilon(u) := J_\varepsilon(u) - \mathcal{K}_\varepsilon(u), \tag{3.1.3}$$

where

$$J_\varepsilon(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \frac{1}{p_\alpha^* + 1 + \varepsilon}\int_\Omega |x|^\alpha u^{p_\alpha^*+1+\varepsilon} \, dx \tag{3.1.4}$$

and

$$\mathcal{K}_\varepsilon(u) := \frac{\lambda_\varepsilon}{2}\int_\Omega |x|^\beta u^2 \, dx.$$

We now took advantage of the fact that the domain has certain symmetries and consider spherical coordinates $x = x(\rho, \boldsymbol{\theta})$ centered at the origin given by $\rho = |x|$ and $\boldsymbol{\theta} = \frac{x}{|x|}$. We also consider the transformation

$$\mathcal{T}(u)(\rho, \boldsymbol{\theta}) = v(\rho, \boldsymbol{\theta}) := \left(\frac{2}{N-2}\right)^{\frac{N-2}{2+\alpha}} e^{-\rho} u\left(e^{-\frac{2}{N-2}\rho}\boldsymbol{\theta}\right), \tag{3.1.5}$$

which is a variation of the Emden-Fowler transformation.

Let $\mathfrak{D} := \left\{(\rho, \boldsymbol{\theta}) : \rho \ge \kappa_0(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in S^{N-1}\right\}$, where $\kappa_0 : S^{N-1} \to \mathbb{R}$ is a continuous function.

After the change of variables (3.1.5), problem (3.1.1) becomes

$$\begin{cases} L(v) = a_\varepsilon e^{\rho\varepsilon}|v|^{p_\alpha^*+\varepsilon} + \lambda_\varepsilon \left(\dfrac{2}{N-2}\right)^2 e^{-\frac{2(\beta+2)}{N-2}\rho} v & \text{in } \mathfrak{D}, \\ v > 0 & \text{in } \mathfrak{D}, \\ v = 0 & \text{on } \partial\mathfrak{D}, \end{cases} \tag{3.1.6}$$

where $a_\varepsilon := \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon}$ and

$$L(v) := -\left(\frac{2}{N-2}\right)^2 \Delta_{S^{N-1}} v - v'' + v. \tag{3.1.7}$$

Here, L is the transformed operator associated with $-\Delta$, where $' = \frac{\partial}{\partial\rho}$ and $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on S^{N-1}.

Proof. Let $r = e^{-\frac{2}{N-2}\rho}$ and $x = x(\rho, \boldsymbol{\theta}) = e^{-\frac{2}{N-2}\rho}\boldsymbol{\theta} \in \Omega$. Then, if we define $x = x(r, \boldsymbol{\theta}) = r\boldsymbol{\theta} \in \Omega$, we have

$$\rho = -\left(\frac{N-2}{2}\right)\ln r$$

and

$$\frac{\partial\rho}{\partial r} = -\left(\frac{N-2}{2}\right)\frac{1}{r}.$$

From (3.1.5), we have

$$u(x(r,\boldsymbol{\theta})) = c\, e^{\rho} v(\rho, \boldsymbol{\theta}),$$

where $c = \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}}$. Moreover, it is know that the Laplacian for these coordinates is defined as

$$\Delta u(x(r,\boldsymbol{\theta})) = \frac{\partial^2 u}{\partial r^2} + \left(\frac{N-1}{r}\right)\frac{\partial u}{\partial r} + \frac{1}{r^2}\Delta_{\boldsymbol{\theta}} u. \tag{3.1.8}$$

Thus, by direct calculations

$$\frac{\partial u}{\partial r} = -c\, e^{\rho}\left(\frac{N-2}{2}\right)\frac{1}{r}v - c\, e^{\rho}\frac{\partial v}{\partial\rho}\left(\frac{N-2}{2}\right)\frac{1}{r},$$

$$\begin{aligned} \frac{\partial^2 u}{\partial r^2} =& c\, e^{\rho}\left(\frac{N-2}{2}\right)^2\frac{1}{r^2}v + c\, e^{\rho}\left(\frac{N-2}{2}\right)\frac{1}{r^2}v + c\, e^{\rho}\left(\frac{N-2}{2}\right)^2\frac{1}{r^2}\frac{\partial v}{\partial\rho} \\ &+ c\, e^{\rho}\left(\frac{N-2}{2}\right)^2\frac{1}{r^2}\frac{\partial v}{\partial\rho} + c\, e^{\rho}\left(\frac{N-2}{2}\right)\frac{1}{r^2}\frac{\partial v}{\partial\rho} + c\, e^{\rho}\frac{\partial^2 v}{\partial\rho^2}\left(\frac{N-2}{2}\right)^2\frac{1}{r^2} \end{aligned}$$

and

$$\Delta_{\boldsymbol{\theta}} u = c\, e^{\rho}\Delta_{\boldsymbol{\theta}} v.$$

Therefore, we can replace in (3.1.8) to get

$$\Delta u(x(r,\boldsymbol{\theta})) = c\, e^{\rho} \left(\frac{N-2}{2}\right)^2 \frac{1}{r^2} \left(-v + \frac{\partial^2 v}{\partial \rho^2} + \left(\frac{2}{N-2}\right)^2 \Delta_{\boldsymbol{\theta}} v\right). \tag{3.1.9}$$

On the other hand, notice that

$$\begin{aligned} |x|^{\alpha} u^{p_{\alpha}^{*}} &= e^{-\frac{2\alpha}{N-2}} \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\left(\frac{N+2\alpha+2}{N-2}+\varepsilon\right)} e^{\frac{N+2\alpha+2}{N-2}\rho} e^{\varepsilon\rho} v^{p_{\alpha}^{*}+\varepsilon} \\ &= e^{\frac{N+2}{N-2}\rho} e^{\varepsilon\rho} \left(\frac{N-2}{2}\right)^{\frac{N+2\alpha+2}{2+\alpha}+\frac{N-2}{2+\alpha}\varepsilon} v^{p_{\alpha}^{*}+\varepsilon} \end{aligned}$$

and

$$\lambda_{\varepsilon} |x|^{\beta} u = \lambda_{\varepsilon} e^{-\frac{2\beta}{N-2}} \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}} e^{\rho} v.$$

Then, replacing all the above in (3.1.1), we conclude that

$$\begin{aligned} c\, e^{\rho} \left(\tfrac{N-2}{2}\right)^2 \frac{1}{e^{-\frac{4}{N-2}}} \left(v - \tfrac{\partial^2 v}{\partial \rho^2} - \left(\tfrac{2}{N-2}\right)^2 \Delta_{\boldsymbol{\theta}} v\right) &= e^{\frac{N+2}{N-2}\rho} e^{\varepsilon\rho} \left(\tfrac{N-2}{2}\right)^{\frac{N+2\alpha+2}{2+\alpha}+\frac{N-2}{2+\alpha}\varepsilon} v^{p_{\alpha}^{*}+\varepsilon} \\ &\quad +\lambda_{\varepsilon} e^{-\frac{2\beta}{N-2}} c e^{\rho} v \\ L(v) &= a_{\varepsilon} e^{\rho\varepsilon} v^{p_{\alpha}^{*}+\varepsilon} + \lambda_{\varepsilon} \left(\tfrac{2}{N-2}\right)^2 e^{-\frac{2(\beta+2)}{N-2}\rho} v. \end{aligned}$$

□

To get a suitable ansatz for (3.1.6), we consider the limit problem (1.2.11) under the transformation $\mathcal{T}$ and its solutions. In other words, we are interested in studying the function

$$W(\rho) := C_{N,\alpha} e^{-\rho} \left(1 + e^{-\frac{4+2\alpha}{N-2}\rho}\right)^{-\frac{N-2}{2+\alpha}},$$

with $\rho \in \mathbb{R}$ and $C_{N,\alpha} := \left(\frac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}}$. Straightforward calculations show that the function W is the unique positive solution of the problem

$$\begin{cases} W'' - W + W^{p_{\alpha}^{*}} = 0 & \text{in } \mathbb{R}, \\ W'(0) = 0, & \\ W(\rho) \to 0 & \text{as } \rho \to \pm\infty. \end{cases}$$

Proof. Indeed, notice that

$$\begin{aligned} W''(\rho) &= W(\rho) + C_{N,\alpha}\, e^{-p_{\alpha}\rho} \left(\tfrac{4(N+\alpha)}{N-2}\right) (1 + e^{-\frac{4+2\alpha}{N-2}\rho})^{-\frac{N-2}{2+\alpha}-2} e^{-\frac{4+2\alpha}{N-2}\rho} \\ &\quad -C_{N,\alpha} \left(\tfrac{4(N+\alpha)}{N-2}\right) e^{-p_{\alpha}\rho} (1 + e^{-\frac{4+2\alpha}{N-2}\rho})^{-\frac{N-2}{2+\alpha}-1} \\ &= W(\rho) + C_{N,\alpha}\, e^{-p_{\alpha}\rho} \left(\tfrac{4(N+\alpha)}{N-2}\right) (1 + e^{-\frac{4+2\alpha}{N-2}\rho})^{-\frac{N-2}{2+\alpha}-2} (e^{-\frac{4+2\alpha}{N-2}\rho} - (1 + e^{-\frac{4+2\alpha}{N-2}\rho})) \\ &= W(\rho) - W^{p_{\alpha}}(\rho). \end{aligned}$$

Moreover

$$\begin{aligned} W'(0) &= 2C_{N,\alpha}(1+1)^{-\frac{N-2}{2+\alpha}-1} - C_{N,\alpha}(1+1)^{-\frac{N-2}{2+\alpha}} \\ &= C_{N,\alpha}\, 2^{-\frac{N-2}{2+\alpha}} - C_{N,\alpha}\, 2^{-\frac{N-2}{2+\alpha}} \\ &= 0. \end{aligned}$$

□

Also, note that if $\mu \neq 1$, then $\mathcal{T}(U_{\mu,\alpha})(\rho,\boldsymbol{\theta}) = W(\rho-\xi)$, for all $\rho \in \mathbb{R}$, where $\xi = -\frac{N-2}{2}\log\mu$.

Proof. By direct calculations, we have

$$\begin{aligned} \mathcal{T}(U_{\mu,\alpha})(\rho,\boldsymbol{\theta}) &= \left(\tfrac{2}{N-2}\right)^{\frac{N-2}{2+\alpha}} e^{-\rho}\, \gamma_{N,\alpha} \left(\frac{\mu^{\frac{2+\alpha}{2}}}{\mu^{2+\alpha}+e^{-\frac{2(2+\alpha)}{N-2}\rho}} \right)^{\frac{N-2}{2+\alpha}} \\ &= \left(\tfrac{4}{(N-2)^2}\right)^{\frac{N-2}{4+2\alpha}} e^{-\rho}\, (N+\alpha)^{\frac{N-2}{4+2\alpha}} (N-2)^{\frac{N-2}{4+2\alpha}} \frac{1}{\mu^{-\frac{N-2}{2}}} \left(\mu^{2+\alpha} + e^{-\frac{2(2+\alpha)}{N-2}\rho} \right)^{-\frac{N-2}{2+\alpha}}. \end{aligned}$$

But $\mu = e^{-\frac{2\xi}{N-2}}$, thus

$$\begin{aligned} W_{\xi,\alpha}(\rho) &= \left(\tfrac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}} e^{-(\rho+\xi)} \left(e^{-\frac{2(2+\alpha)}{N-2}\xi} + e^{-\frac{2(2+\alpha)}{N-2}\rho} \right)^{-\frac{N-2}{2+\alpha}} \\ &= \left(\tfrac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}} e^{-(\rho+\xi)} e^{2\xi} \left(1 + \frac{e^{-\frac{2(2+\alpha)}{N-2}\rho}}{e^{-\frac{2(2+\alpha)}{N-2}\xi}} \right)^{-\frac{N-2}{2+\alpha}} \\ &= \left(\tfrac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}} e^{-(\rho-\xi)} \left(1 + e^{-\frac{2(2+\alpha)}{N-2}(\rho-\xi)} \right)^{-\frac{N-2}{2+\alpha}} \\ &= W(\rho-\xi). \end{aligned}$$

□

Another point we must consider is the boundary condition of (3.1.6), as W does not satisfy it. To address this, we introduce the function $\Pi_{\xi,\alpha} : \mathfrak{D} \to \mathbb{R}$, defined as $\Pi_{\xi,\alpha} = \mathcal{T}(\pi_{\mu,\alpha})$. Then, $\Pi_{\xi,\alpha}$ solves the problem

$$\begin{cases} L(\Pi_{\xi,\alpha}) = 0 & \text{in } \mathfrak{D}, \\ \Pi_{\xi,\alpha} = -W(\cdot - \xi) & \text{on } \partial\mathfrak{D}. \end{cases}$$

Lastly, for each $i = 1, 2, \ldots, k$, let ξ_i be a positive value such that $0 < \xi_1 < \xi_2 < \ldots < \xi_k$ and $\xi_i = -\frac{N-2}{2}\log\mu_i$. We seek a solution to problem (3.1.6) of the form

$$v_{k,\varepsilon}(\rho,\boldsymbol{\theta}) = \sum_{i=1}^{k} \left(W(\rho-\xi_i) + \Pi_{\xi_i,\alpha}(\rho,\boldsymbol{\theta}) \right) + \phi(\rho,\boldsymbol{\theta}), \quad (\rho,\boldsymbol{\theta}) \in \mathfrak{D}, \tag{3.1.10}$$

where $\phi : \mathfrak{D} \to \mathbb{R}$ is a function of small order that is symmetric with respect to the variables $\theta_1, \theta_2, \ldots, \theta_N$. Note that $v_{k,\varepsilon}$ solves (3.1.6) if and only if $u_{k,\varepsilon}$ given by (1.4.4) solves (3.1.1). Therefore, for large values of the ξ_is, the ansatz presented for $v_{k,\varepsilon}$ provides, after returning to the original variables, a solution with the shape of a tower of *bubbles of order* α for (3.1.1).

3.2 Expansion of the reduced energy

Notice that the energy functional associated with problem (3.1.6) is

$$E_\epsilon(v) := I_\epsilon(v) - K_\epsilon(v), \tag{3.2.1}$$

where

$$I_\varepsilon(v) := \frac{1}{2}\int_{\mathfrak{D}} \left(\left(\frac{2}{N-2}\right)^2 |\nabla_{\boldsymbol{\theta}} v|^2 + |v'|^2 + |v|^2\right) d\rho\, d\boldsymbol{\theta} - \frac{a_\varepsilon}{p_\alpha^* + 1 + \varepsilon}\int_{\mathfrak{D}} e^{\varepsilon\rho}|v|^{p_\alpha^*+1+\varepsilon}\, d\rho\, d\boldsymbol{\theta} \tag{3.2.2}$$

and

$$K_\epsilon(v) := \frac{\lambda_\varepsilon}{2}\left(\frac{2}{N-2}\right)^2 \int_{\mathfrak{D}} e^{-\frac{2(\beta+2)}{N-2}\rho}|v|^2\, d\rho\, d\boldsymbol{\theta}. \tag{3.2.3}$$

Note that

$$I_\varepsilon(v) = -\left(\frac{2}{N-2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1} J_\varepsilon(u), \tag{3.2.4}$$

where $v = \mathcal{T}(u)$ and J_ε is given by (3.1.4).

Proof. Indeed, consider first

$$\frac{1}{p_\alpha + 1 + \varepsilon}\int_\Omega |x|^\alpha u^{p_\alpha+1+\varepsilon} dx.$$

Using $\mathcal{T}$ defined in (3.1.5), we have

$$|x|^\alpha u^{p_\alpha+1+\varepsilon} = e^{-\frac{2\alpha}{N-2}\rho}\left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\cdot\frac{2(N+\alpha)}{N-2}} \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\frac{2(N+\alpha)}{N-2}\rho}\, e^{\rho\varepsilon}\, |v|^{p_\alpha+1+\varepsilon}.$$

Next, we can integrate

$$\begin{aligned}
&\tfrac{1}{p_\alpha+1+\varepsilon}\int_\Omega |x|^\alpha u^{p_\alpha+1+\varepsilon}\, dx \\
&\quad = \tfrac{1}{p_\alpha+1+\varepsilon}\int_{\mathcal{Q}} e^{\frac{2N}{N-2}\rho}\left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}} \left(\tfrac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\rho\varepsilon}|v|^{p_\alpha+1+\varepsilon}\, d\rho\, d\boldsymbol{\theta} \\
&\quad = \tfrac{1}{p_\alpha+1+\varepsilon}\int_{\mathcal{Q}} e^{\frac{2N}{N-2}\rho}\left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}} \left(\tfrac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\rho\varepsilon}|v|^{p_\alpha+1+\varepsilon} e^{-\frac{2(N-1)}{N-2}\rho}\left(-\tfrac{2}{N-2}\right) e^{-\frac{2}{N-2}\rho}\, d\rho\, d\boldsymbol{\theta} \\
&\quad = -\tfrac{1}{p_\alpha+1+\varepsilon}\left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1}\int_{\mathcal{Q}} \left(\tfrac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\rho\varepsilon}|v|^{p_\alpha+1+\varepsilon}\, d\rho\, d\boldsymbol{\theta},
\end{aligned}$$

to conclude that

$$-\tfrac{1}{p_\alpha+1+\varepsilon}\left(\tfrac{2}{N-2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1}\int_\Omega |x|^\alpha u^{p_\alpha+1+\varepsilon}\, dx = \tfrac{1}{p_\alpha+1+\varepsilon}\int_{\mathcal{Q}} \left(\tfrac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\rho\varepsilon}|v|^{p_\alpha+1+\varepsilon}\, d\rho\, d\boldsymbol{\theta}.$$

Even more, by straightforward calculations

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dx \\
&= \frac{1}{2}\int_{\Omega} \left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}} e^{\frac{2N}{N-2}\rho}\left(v^2 + \left|\frac{\partial v}{\partial \rho}\right|^2 + \left(\tfrac{2}{N-2}\right)^2 |\nabla_{\boldsymbol{\theta}} v|^2\right) dx \\
&= \frac{1}{2}\int_{\mathcal{Q}} \left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}} e^{\frac{2N}{N-2}\rho}\left(v^2 + \left|\frac{\partial v}{\partial \rho}\right|^2 + \left(\tfrac{2}{N-2}\right)^2 |\nabla_{\boldsymbol{\theta}} v|^2\right) e^{-\frac{2(N-1)}{N-2}\rho}\left(-\tfrac{2}{N-2}\right) e^{-\frac{2}{N-2}\rho} \, d\rho \, d\boldsymbol{\theta} \\
&= -\left(\tfrac{N-2}{2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1} \frac{1}{2}\int_{\mathcal{Q}} \left(v^2 + \left|\frac{\partial v}{\partial \rho}\right|^2 + \left(\frac{2}{N-2}\right)^2 |\nabla_{\boldsymbol{\theta}} v|^2\right) d\rho \, d\boldsymbol{\theta},
\end{aligned}$$

which let us conclude that

$$-\left(\frac{2}{N-2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1} \frac{1}{2}\int_{\Omega} |\nabla u|^2 \, dx = \frac{1}{2}\int_{\mathcal{Q}} \left(v^2 + \left|\frac{\partial v}{\partial \rho}\right|^2 + \left(\frac{2}{N-2}\right)^2 |\nabla_{\boldsymbol{\theta}} v|^2\right) d\rho \, d\boldsymbol{\theta},$$

ending the proof. □

Since we are looking for a solution $v_{k,\varepsilon}$, for $i = 1, \ldots, k$ and positive values $\xi_i \in \mathbb{R}$, we will use the following notation.

$$W_i(\rho) := W(\rho - \xi_i), \quad \Pi_i := \Pi_{\xi_i,\alpha}, \quad V_i := W_i + \Pi_i, \quad V := \sum_{i=1}^{k} V_i. \tag{3.2.5}$$

Here we have chosen the ξ_is conveniently as

$$\xi_1 = -\frac{1}{2}\log\varepsilon + \log\mu_1, \quad \xi_{i+1} - \xi_i = -\log\varepsilon - \log\mu_{i+1} \quad \text{for } i = 1, 2, \ldots, k-1, \tag{3.2.6}$$

where the μ_is are positive parameters to be determined.

Before stating the main lemma of this chapter, we need the following proposition.

Proposition 3.2.1. *The functions defined in (3.2.5) satisfy the following estimates*

$$|W_i(\rho)| \le Ce^{-|\rho-\xi_i|}, \quad |\Pi_i(\rho)| \le Ce^{-|\rho-\xi_i|} \quad \textit{and} \quad |V_i(\rho)| \le Ce^{-|\rho-\xi_i|},$$

where C is a positive constant depending only on N and α.

Also, in the remainder of this paper, ω_{N-1} denotes the surface area of S^{N-1}.

Aiming at our goal of proving that $v_{k,\varepsilon}$ is a solution of (3.1.6), let us proceed to estimate $E_\varepsilon(V)$.

Lemma 3.2.2. *Let $k \in \mathbb{N}$, $k \ge 2$, let $\delta > 0$, and assume that*

$$\delta < \mu_i < \delta^{-1} \quad \textit{for } i = 1, 2, \ldots, k. \tag{3.2.7}$$

In addition, assume that $\lambda_\varepsilon = \eta\varepsilon^{\frac{N-4-\beta}{N-2}}$, for some $\eta > 0$ sufficiently large. Then, for V defined by (3.2.5) and points ξ_is as in (3.2.6), there are positive numbers a_1, a_2, a_3, a_4, a_5 and a_6, which depend *only on N and α, such that*

$$E_\varepsilon(V) = ka_1 + \varepsilon\Psi_k(\boldsymbol{\mu}) + \frac{k^2}{2}\left(\varepsilon\log\varepsilon\right)a_2 + k\varepsilon a_4 + \varepsilon\Theta_\varepsilon(\boldsymbol{\mu}),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$,

$$\Psi_k(\boldsymbol{\mu}) = a_2\sum_{i=2}^{k}(k-i+1)\log\mu_i - a_3\sum_{i=2}^{k}\mu_i + a_5\frac{1}{\mu_1^2} - k\log\mu_1 - a_6\eta\mu_1^{-2\frac{(\beta+2)}{N-2}} \tag{3.2.8}$$

and $\Theta_\varepsilon(\boldsymbol{\mu}) \to 0$ as $\varepsilon \to 0$ uniformly with respect to the values μ_i satisfying (3.2.7). Here,

$$a_1 := \left(\frac{2}{N-2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1}\left(\frac{2+\alpha}{2(N+\alpha)}\right)\int_{\mathbb{R}^N}|x|^\alpha U_{1,\alpha}^{p_\alpha^*+1}dx, \tag{3.2.9}$$

$$a_2 := \frac{\omega_{N-1}}{p_\alpha^*+1}\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho, \tag{3.2.10}$$

$$a_3 := \left(\frac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{4+2\alpha}}\omega_{N-1}\int_{\mathbb{R}} e^{|\rho|}W^{p_\alpha^*}d\rho, \tag{3.2.11}$$

$$a_4 := \frac{\omega_{N-1}}{p_\alpha^*+1}\left(\left(\frac{1}{p_\alpha^*+1} - \log\left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}}\right)\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho - \int_{\mathbb{R}} W^{p_\alpha^*+1}\log W\,d\rho\right), \tag{3.2.12}$$

$$a_5 := \frac{1}{2}\left(\frac{2}{N-2}\right)^{\frac{2(N+\alpha)}{2+\alpha}-1}\left((N+\alpha)(N-2)\right)^{\frac{N-2}{4+2\alpha}}H(0,0)\int_{\mathbb{R}^N}|x|^\alpha U_{1,\alpha}^{p_\alpha^*}dx \tag{3.2.13}$$

and

$$a_6 := \omega_{N-1}\frac{1}{2}\left(\frac{2}{N-2}\right)^2\int_{\mathbb{R}} e^{-\frac{2(\beta+2)}{N-2}\rho}W^2\,d\rho. \tag{3.2.14}$$

Proof. First, we combine (3.2.6) with (3.2.7) and obtain

$$\xi_1 > -\log\left(\frac{\sqrt{\varepsilon}}{\delta}\right) \quad \text{and} \quad \xi_{i+1} - \xi_i > -\log\left(\frac{\varepsilon}{\delta}\right) \quad \text{for } i = 1, 2, \ldots, k-1.$$

We also consider the numbers

$$\zeta_1 = 0, \quad \zeta_i = \frac{\xi_{i-1}+\xi_i}{2} \quad \text{for all } i = 2, \ldots, k, \quad \zeta_{k+1} = \infty, \tag{3.2.15}$$

and the sets $\mathfrak{D}_i = \{(\rho, \boldsymbol{\theta}) \in \mathfrak{D} : \zeta_i \le \rho \le \zeta_{i+1}\}$ for all $i = 1, 2, \ldots, k$.

We now make some basic estimates. Let $l \in \{1, 2, \ldots, k\}$ be fixed. Then, it is not difficult to check that for every $i, j \in \{1, 2, \ldots, k\}$, one has

$$\int_{\mathfrak{D}_l} W_i^{p_\alpha^*}W_j\,d\rho\,d\boldsymbol{\theta} = o(\varepsilon^{\frac{p_\alpha^*+1}{2}}) \quad \text{if } i \neq l. \tag{3.2.16}$$

Indeed, if $i = j$ and $i > l$, we can use Proposition 3.2.1 to get

$$\begin{aligned}\int_{\mathfrak{D}_l} W_i^{p_\alpha^*+1} d\rho d\boldsymbol{\theta} \le\ & C\omega_{N-1}\int_{\zeta_l}^{\zeta_{l+1}} e^{-(p_\alpha^*+1)|\rho-\xi_i|}\,d\rho \\ =\ & Ce^{-(p_\alpha^*+1)(\rho-\xi_i)}\Big|_{\zeta_l}^{\zeta_{l+1}} \\ \le\ & Ce^{-(p_\alpha^*+1)(\zeta_l-\xi_i)} \\ \le\ & Ce^{-\frac{p_\alpha^*+1}{2}(\xi_{i+1}-\xi_i)} \\ \le\ & C\varepsilon^{\frac{p_\alpha^*+1}{2}} = o(\varepsilon^{\frac{p_\alpha^*+1}{2}})\end{aligned}$$

and the case $i < l$ is analogous. Moreover, for every $j \in \{1, 2, \dots, k\}$, if $i \neq j$, we have

$$\begin{aligned}\int_{\mathfrak{D}_l} W_i^{p_\alpha^*} W_j\, d\rho\, d\boldsymbol{\theta} =\ & \omega_{N-1}\int_{\zeta_l}^{\zeta_{l+1}} W_i^{p_\alpha^*} W_j\, d\rho \\ \le\ & C\int_{\zeta_l}^{\zeta_{l+1}} e^{-p_\alpha^*|\rho-\xi_i|} e^{-|\rho-\xi_j|}\, d\rho \\ =\ & C\int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-p_\alpha^*|\rho|} e^{-|\rho+\xi_i-\xi_j|}\, d\rho \\ \le\ & C\int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-p_\alpha^*|\rho|} e^{|\rho|-|\xi_i-\xi_j|}\, d\rho \\ =\ & Ce^{-|\xi_i-\xi_j|}\int_{\zeta_l-\xi_i}^{\zeta_{l+1}-\xi_i} e^{-(p_\alpha^*-1)|\rho|}\, d\rho \\ \le\ & C\varepsilon\int_{\zeta_l}^{\zeta_{l+1}} e^{-(p_\alpha^*-1)|\rho-\xi_i|}\, d\rho \\ \le\ & C\varepsilon\,\varepsilon^{\frac{p-1}{2}} \\ \le\ & C\varepsilon^{\frac{p+1}{2}}.\end{aligned}$$

On the other hand, if $i = l$, then

$$\int_{\mathfrak{D}_l} W_l^{p_\alpha^*} W_j\, d\rho\, d\boldsymbol{\theta} = \omega_{N-1} C_{N,\alpha} e^{-|\xi_l-\xi_j|}\int_{\mathbb{R}} e^{|\rho|} W^{p_\alpha^*}\, d\rho + o(\varepsilon) = o(\varepsilon) \tag{3.2.17}$$

if $i = j$ and $j \neq l$, since W is even. Indeed, notice that

$$\begin{aligned}\int_{\mathfrak{D}_l} W_l^{p_\alpha^*} W_j\, d\rho\, d\boldsymbol{\theta} =\ & \omega_{N-1} C_{N,\alpha}(1+o(1))\int_{\zeta_l}^{\zeta_{l+1}} W_l^{p_\alpha^*} e^{-|\rho-\xi_j|}\, d\rho \\ =\ & \omega_{N-1} C_{N,\alpha}(1+o(1))\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} W^{p_\alpha^*} e^{|\rho|-|\xi_l-\xi_j|}\, d\rho \\ =\ & \omega_{N-1} C_{N,\alpha} e^{-|\xi_l-\xi_j|}\int_{\mathbb{R}} W^{p_\alpha^*} e^{|\rho|}\, d\rho + o(\varepsilon) \\ =\ & C\varepsilon + o(\varepsilon) \\ =\ & o(\varepsilon).\end{aligned}$$

Now, by setting

$$\overline{W}=\sum_{i=1}^{k}W_i \quad \text{and} \quad \overline{\Pi}=\sum_{i=1}^{k}\Pi_i,$$

from the definition of V in (3.2.5), we obtain $\overline{W}=V-\overline{\Pi}$. Then, by Proposition 3.2.1

$$0\le W_i-V_i=-\Pi_i\le Ce^{-\xi_i}\le C\varepsilon^{\frac{1}{2}} \quad \text{in } \mathfrak{D},$$

for all $i=1,2,\dots,k$. Let $h(t)=|\overline{W}+t\overline{\Pi}|^{p_\alpha^*+1}$. By applying the Mean value Theorem, we estimate

$$\begin{aligned}\left|\int_{\mathfrak{D}}\left(\overline{W}^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\, d\boldsymbol{\theta}\right| &= \left|\int_{\mathfrak{D}}h(0)-h(1)\, d\rho\, d\boldsymbol{\theta}\right|\\ &= (p_\alpha^*+1)\int_{\mathfrak{D}}|\overline{W}+t\overline{\Pi}|^{p_\alpha^*}\,\overline{\Pi}\, d\rho\, d\boldsymbol{\theta}\,(0-1)\\ &\le (p_\alpha^*+1)\int_{\mathfrak{D}}\left(\sum_{i=1}^{k}W_i\right)^{p_\alpha^*}|\overline{\Pi}|\, d\rho\, d\boldsymbol{\theta}\\ &= o(\varepsilon).\end{aligned} \tag{3.2.18}$$

In the same way, letting $g(t)=\left|W_l+t\sum_{j\neq l}^{k}W_j\right|^{p_\alpha^*+1}$, we can use the Mean value Theorem to estimate

$$\begin{aligned}\int_{\mathfrak{D}}\left(\sum_{l=1}^{k}W_l^{p_\alpha^*+1}-\overline{W}^{p_\alpha^*+1}\right)d\rho\, d\boldsymbol{\theta} &= \sum_{l=1}^{k}\int_{\mathfrak{D}_l}\left(W_l^{p_\alpha^*+1}-|W_l+\sum_{j\neq l}W_j|^{p_\alpha^*+1}\right)d\rho\, d\boldsymbol{\theta}+o(\varepsilon)\\ &= \sum_{l=1}^{k}\int_{\mathfrak{D}_l}g(0)-g(1)\, d\rho\, d\boldsymbol{\theta}+o(\varepsilon)\\ &\le (p_\alpha^*+1)\sum_{l=1}^{k}\int_{\mathfrak{D}_l}\left(\sum_{i=l}^{k}W_i\right)^{p_\alpha^*}\left(\sum_{j\neq l}^{k}W_j\right)(-1)\, d\rho\, d\boldsymbol{\theta}+o(\varepsilon)\\ &\le C\sum_{i,l=1}^{k}\int_{\mathfrak{D}_l}W_i^{p_\alpha^*}\left(\sum_{j\neq l}^{k}W_j\right)d\rho\, d\boldsymbol{\theta}+o(\varepsilon)\\ &\le Co(\varepsilon)+o(\varepsilon)\\ &= o(\varepsilon).\end{aligned} \tag{3.2.19}$$

From (3.2.18) and (3.2.19), we obtain

$$\int_{\mathfrak{D}}V^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta}=\sum_{l=1}^{k}\int_{\mathfrak{D}_l}W_l^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta}+o(1)=k\,\omega_{N-1}\int_{\mathbb{R}}W^{p_\alpha^*+1}d\rho+o(1).$$

Next, we are interested in estimate $I_\varepsilon(V)$. Note that

$$I_\varepsilon(V)=I_0(V)+A_{1,\varepsilon}+A_{2,\varepsilon}+A_{3,\varepsilon}\,, \tag{3.2.20}$$

where

$$I_0\left(V\right) = \frac{1}{2}\int_{\mathfrak{D}}\left(\left(\frac{2}{N-2}\right)^2|\nabla_\theta V|^2 + |V'|^2 + V^2\right)d\rho\, d\boldsymbol{\theta} - \frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}} V^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta},$$

$$A_{1,\varepsilon} = \left(\frac{1}{p_\alpha^*+1} - \frac{1}{p_\alpha^*+1+\varepsilon}\right)\int_{\mathfrak{D}} V^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta},$$

$$A_{2,\varepsilon} = \frac{1}{p_\alpha^*+1+\varepsilon}\int_{\mathfrak{D}}\left(V^{p_\alpha^*+1} - V^{p_\alpha^*+1+\varepsilon}\right)d\rho\, d\boldsymbol{\theta}$$

and

$$A_{3,\varepsilon} = \frac{1}{p_\alpha^*+1+\varepsilon}\int_{\mathfrak{D}}\left(1 - a_\varepsilon e^{\varepsilon\rho}\right)V^{p_\alpha^*+1+\varepsilon}d\rho\, d\boldsymbol{\theta}.$$

If we define $q(\varepsilon) = \left(\frac{1}{p_\alpha^*+1} - \frac{1}{p_\alpha^*+1+\varepsilon}\right)|V|^{p_\alpha^*+1}$ and $r(\varepsilon) = \frac{1}{p_\alpha^*+1+\varepsilon}\left(|V|^{p_\alpha^*+1} - |V|^{p_\alpha^*+1+\varepsilon}\right)$, it is not difficult to verify, using a Taylor expansion and the Mean value Theorem, that

$$A_{1,\varepsilon} = \frac{k\,\omega_{N-1}\varepsilon}{\left(p_\alpha^*+1\right)^2}\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho + o(\varepsilon), \tag{3.2.21}$$

$$A_{2,\varepsilon} = -\frac{k\,\omega_{N-1}\varepsilon}{p_\alpha^*+1}\int_{\mathbb{R}} W^{p_\alpha^*+1}\log W\, d\rho + o(\varepsilon). \tag{3.2.22}$$

respectively. For $A_{3,\varepsilon}$, we define $s(\varepsilon) = \frac{V^{p_\alpha^*+1+\varepsilon}}{p_\alpha^*+1+\varepsilon}\left(1 - \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}\varepsilon} e^{\rho\varepsilon}\right)$ and use (3.2.6) to get

$$\begin{aligned}
A_{3,\varepsilon} =&\ \frac{1}{p_\alpha^*+1+\varepsilon}\int_{\mathfrak{D}}\left(1 - a_\varepsilon e^{\varepsilon\rho}\right)V^{p_\alpha^*+1+\varepsilon}d\rho\, d\boldsymbol{\theta}\\
=&\ \frac{1}{p_\alpha^*+1+\varepsilon}\int_{\mathfrak{D}}\left(1 - a_\varepsilon e^{\varepsilon\rho}\right)W^{p_\alpha^*+1+\varepsilon}d\rho\, d\boldsymbol{\theta} + o(\varepsilon)\\
=&\ -\frac{\varepsilon}{p_\alpha^*+1}\left(\frac{N-2}{2+\alpha}\log\left(\frac{N-2}{2}\right)\right)\int_{\mathfrak{D}} W^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta} - \frac{\varepsilon}{p_\alpha^*+1}\int_{\mathfrak{D}}\rho W^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta} + o(\varepsilon)\\
=&\ -\frac{\varepsilon}{p_\alpha^*+1}\left(\frac{N-2}{2+\alpha}\log\left(\frac{N-2}{2}\right)\right)\int_{\mathfrak{D}} W^{p_\alpha^*+1}d\rho\, d\boldsymbol{\theta} - \frac{\varepsilon}{p_\alpha^*+1}\sum_{l=1}^{k}\int_{\mathfrak{D}_l}\rho W^{p_\alpha^*+1}(\rho-\xi_l)\, d\rho\, d\boldsymbol{\theta} + o(\varepsilon)\\
=&\ -\frac{\omega_{N-1}\varepsilon}{p_\alpha^*+1}\left(\frac{N-2}{2+\alpha}\log\left(\frac{N-2}{2}\right)\right)\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho\\
&\ -\frac{\omega_{N-1}\varepsilon}{p_\alpha^*+1}\sum_{l=1}^{k}\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}}(y+\xi_l)W^{p_\alpha^*+1}(y)\, dy + o(\varepsilon)\\
=&\ -\frac{\omega_{N-1}\varepsilon}{p_\alpha^*+1}\left[\left(\frac{N-2}{2+\alpha}\log\left(\frac{N-2}{2}\right)\right) + \left(\sum_{l=1}^{k}\xi_l\right)\right]\int_{\mathbb{R}} W^{p_\alpha^*+1}(\rho)d\rho + o(\varepsilon)\\
=&\ -\frac{\omega_{N-1}\varepsilon}{p_\alpha^*+1}\left(\log\left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}}\right)\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho\\
&\ +\frac{\omega_{N-1}\varepsilon}{p_\alpha^*+1}\left(\frac{k^2}{2}\log\varepsilon - k\log\mu_1 + \sum_{i=2}^{k}(k-i+1)\log\mu_i\right)\int_{\mathbb{R}} W^{p_\alpha^*+1}d\rho + o(\varepsilon).
\end{aligned} \tag{3.2.23}$$

On the other hand, notice that

$$I_0\left(V\right)-\sum_{i=1}^{k} I_0\left(V_i\right)=\frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}}\!\!\left(\sum_{i=1}^{k} V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\,d\boldsymbol{\theta}+\sum_{i=2}^{k}\sum_{j=1,j<i}^{k-1}\int_{\mathfrak{D}}\!W_i^{p_\alpha^*}V_j\,d\rho\,d\boldsymbol{\theta}.$$

Indeed,

$$\begin{aligned}
I_0(V)-\sum_{i=1}^{k} I_0(V_i)=&\ \frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}}\left(\sum_{i=1}^{k} V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\,d\boldsymbol{\theta}\\
&+\frac{1}{2}\int_{\mathfrak{D}}\left(\left(\frac{2}{N-2}\right)^2|\nabla_{\boldsymbol{\theta}}V|^2+|V'|^2+V^2\right)d\rho\,d\boldsymbol{\theta}\\
&-\frac{1}{2}\int_{\mathfrak{D}}\left(\sum_{i=1}^{k}\left(\frac{2}{N-2}\right)^2|\nabla_{\boldsymbol{\theta}}V_i|^2+|V_i'|^2+V_i^2\right)d\rho\,d\boldsymbol{\theta}\\
=&\ \frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}}\left(\sum_{i=1}^{k} V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\,d\boldsymbol{\theta}\\
&+\sum_{i=1}^{k}\sum_{j<i}^{k}\int_{\mathfrak{D}}\left(\left(\frac{2}{N-2}\right)^2\nabla_{\boldsymbol{\theta}}V_i\nabla_{\boldsymbol{\theta}}V_j+V_i'V_j'+V_iV_j\right)d\rho\,d\boldsymbol{\theta}\\
=&\ \frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}}\left(\sum_{i=1}^{k} V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\,d\boldsymbol{\theta}\\
&+\sum_{i=1}^{k}\sum_{j<i}^{k}\int_{\mathfrak{D}}\left(-\left(\frac{2}{N-2}\right)^2\Delta_{\boldsymbol{\theta}}V_i-V_i''+V_i\right)V_j\,d\rho\,d\boldsymbol{\theta}.
\end{aligned}$$

But $W''-W+W^{p_\alpha}=0$ and $\Pi''-\Pi=0$, therefore

$$I_0(V)-\sum_{i=1}^{k} I_0(V_i)=\frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}}\left(\sum_{i=1}^{k} V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}\right)d\rho\,d\boldsymbol{\theta}+\sum_{i=1}^{k}\sum_{j<i}^{k}\int_{\mathfrak{D}}W_i^{p_\alpha^*}V_j\,d\rho\,d\boldsymbol{\theta}$$

and in this way, we can rearrange the previous expression as

$$I_0\left(V\right)=\sum_{i=1}^{k} I_0\left(V_i\right)+\sum_{l=1}^{k}\left(B_{1,l}+B_{2,l}+B_{3,l}\right),$$

where, for every $l=1,2,\ldots,k$, one has

$$B_{1,l}=\frac{1}{p_\alpha^*+1}\int_{\mathfrak{D}_l}\sum_{i=1}^{k}\left(V_i^{p_\alpha^*+1}-V^{p_\alpha^*+1}+\left(p_\alpha+1\right)V_i^{p_\alpha^*}\sum_{j=1,j\neq l}^{k}V_j\right)d\rho\,d\boldsymbol{\theta},$$

$$B_{2,l}=-\int_{\mathfrak{D}_l}\sum_{i=1}^{k}\left(W_i^{p_\alpha^*}\sum_{j=1,j>l}^{k}V_j\right)d\rho\,d\boldsymbol{\theta}$$

and

$$B_{3,l} = \int_{\mathfrak{D}_l} \sum_{i=1}^{k} \bigg(\sum_{j=1, j\neq l}^{k} \Big(W_i^{p_\alpha^*} - V_i^{p_\alpha^*} \Big) V_j \bigg) d\rho \, d\boldsymbol{\theta}.$$

We will prove that, for every $l = 1, 2, \dots, k$, $B_{1,l}$ and $B_{3,l}$ are of lower order than $B_{2,l}$. For $B_{1,l}$, we can use the Mean value Theorem twice to get

$$\begin{aligned}
B_{1,l} =&\ \frac{1}{p_\alpha^* + 1} \int_{\mathfrak{D}_l} \sum_{i=1}^{k} \bigg(V_l^{p_\alpha^*+1} - V^{p_\alpha^*+1} + (p_\alpha + 1) \, V_l^{p_\alpha^*} \sum_{j=1, j\neq l}^{k} V_j \bigg) d\rho \, d\boldsymbol{\theta} + o(\varepsilon) \\
=&\ \omega_{N-1} \int_{\zeta_l}^{\zeta_{l+1}} \bigg(\Big(- (V_l + \vartheta \sum_{j=1, j\neq l}^{k} V_j)^{p_\alpha^*} + V_l^{p_\alpha^*} \Big) \sum_{j=1, j\neq l}^{k} V_j \bigg) \, d\rho + o(\varepsilon) \\
=&\ p_\alpha^* \omega_{N-1} \int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1, j\neq l}^{k} V_j \bigg(V_l + \tau\vartheta \sum_{j=1, j\neq l}^{k} V_j \bigg)^{p_\alpha^*-1} \vartheta \sum_{j=1, j\neq l}^{k} V_j \, d\rho + o(\varepsilon) \\
\leq&\ C \int_{\zeta_l}^{\zeta_{l+1}} \bigg(\sum_{j=1, j\neq l}^{k} V_j \bigg)^2 \bigg(\sum_{j=1}^{k} V_j \bigg)^{p_\alpha^*-1} d\rho + o(\varepsilon) \\
\leq&\ C \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p_\alpha^*-1)|\rho - \xi_l|} \bigg(\sum_{j=1, j\neq l}^{k} V_j \bigg)^2 d\rho + o(\varepsilon) \\
=&\ C \int_{\zeta_l}^{\zeta_{l+1}} e^{-(p_\alpha^*-1)|\rho - \xi_l|} \bigg(\sum_{j=1, j\neq l}^{k} V_j \bigg)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \bigg(\sum_{j=1, j\neq l}^{k} V_j \bigg)^{\frac{p_\alpha^*+1}{p_\alpha^*}} d\rho + o(\varepsilon) \\
\leq&\ C \left(\int_{\zeta_l}^{\zeta_{l+1}} e^{-p_\alpha^*|\rho - \xi_l|} \sum_{j=1, j\neq l}^{k} V_j \, d\rho \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \left(\int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1, j\neq l}^{k} V_j \, d\rho \right)^{\frac{p_\alpha^*+1}{p_\alpha^*}} + o(\varepsilon) \\
\leq&\ C \left(\int_{\zeta_l}^{\zeta_{l+1}} e^{-p_\alpha^*|\rho - \xi_l|} \sum_{j=1, j\neq l}^{k} e^{-|\rho - \xi_j|} \, d\rho \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \left(\int_{\zeta_l}^{\zeta_{l+1}} \sum_{j=1, j\neq l}^{k} V_j^{p_\alpha^*+1} d\rho \right)^{\frac{1}{p_\alpha^*}} + o(\varepsilon) \\
\leq&\ C \left(\int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} e^{-p_\alpha^*|\rho|} \sum_{j=1, j\neq l}^{k} e^{|\rho| - |\xi_l - \xi_j|} \, d\rho \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \varepsilon^{\frac{p_\alpha^*+1}{2p_\alpha^*}} + o(\varepsilon) \\
\leq&\ C \left(\varepsilon \int_{\mathbb{R}} e^{-(p_\alpha^*-1)|\rho|} \, d\rho \right)^{\frac{p_\alpha^*-1}{p_\alpha^*}} \varepsilon^{\frac{p_\alpha^*+1}{2p_\alpha^*}} + o(\varepsilon) \\
=&\ C \varepsilon^{\frac{p_\alpha^*-1}{p_\alpha^*}} \varepsilon^{\frac{p_\alpha^*+1}{2p_\alpha^*}} + o(\varepsilon) \\
=&\ o(\varepsilon)
\end{aligned}$$

and this holds for every $l = 1, 2, \dots, k$.

For $B_{2,l}$, we need to separate the cases $j \geq l+2$ and $j = l+1$. The first case implies that $-|\xi_l - \xi_j| \leq 2\log(\varepsilon) - C$ with $C \geq 0$. Then, for every $l = 1, 2, \dots, k$

$$\begin{aligned}
B_{2,l} = &\ -\omega_{N-1} \sum_{j=l+2}^{k} \int_{\zeta_l}^{\zeta_{l+1}} W_l^{p_\alpha^*} V_j \, d\rho + o(\varepsilon) \\
\leq &\ \omega_{N-1} \, C_{N,\alpha} \sum_{j=l+2}^{k} \int_{\zeta_l}^{\zeta_{l+1}} W_l^{p_\alpha^*} e^{-|\rho - \xi_j|} \, d\rho + o(\varepsilon) \\
\leq &\ \omega_{N-1} \, C_{N,\alpha} \sum_{j=l+2}^{k} \int_{\frac{\xi_{l-1}-\xi_l}{2}}^{\frac{\xi_{l+1}-\xi_l}{2}} W^{p_\alpha^*} e^{|\rho| - |\xi_l - \xi_j|} \, d\rho + o(\varepsilon) \\
\leq &\ \omega_{N-1} \, C_{N,\alpha} \sum_{j=l+2}^{k} e^{-|\xi_l - \xi_j|} \int_{\mathbb{R}} e^{|\rho|} W^{p_\alpha^*} \, d\rho + o(\varepsilon) \\
\leq &\ C\varepsilon^2 + o(\varepsilon) \\
= &\ o(\varepsilon)
\end{aligned}$$

and such, the only relevant term is the case $j = l+1$. In this case, $B_{2,l}$ can be estimate as

$$B_{2,l} = -\omega_{N-1} \, C_{N,\alpha} e^{-(\xi_{l+1} - \xi_l)} \int_{\mathbb{R}} e^{|\rho|} W^{p_\alpha^*} \, d\rho + o(\varepsilon).$$

Lastly, to estimate $B_{3,l}$, we can use the Mean value Theorem to obtain

$$\begin{aligned}
B_{3,l} = &\ \int_{\mathfrak{D}_l} \sum_{i=1}^{k} \left(\sum_{j=1, j \neq l}^{k} \left(W_i^{p_\alpha^*} - V_i^{p_\alpha^*} \right) V_j \right) d\rho \, d\boldsymbol{\theta} \\
= &\ \omega_{N-1} \sum_{j=1, j \neq l}^{k} \int_{\zeta_l}^{\zeta_{l+1}} \left(W_l^{p_\alpha^*} - V_l^{p_\alpha^*} \right) V_j \, d\rho + o(\varepsilon) \\
\leq &\ C \sum_{j=1, j \neq l}^{l-1} \int_{\zeta_l}^{\zeta_{l+1}} \Pi_l V_j \, d\rho + o(\varepsilon) \\
\leq &\ C e^{-\xi_l} + o(\varepsilon) \\
\leq &\ C \varepsilon^{\frac{3}{2}} + o(\varepsilon) \\
= &\ o(\varepsilon),
\end{aligned}$$

for every $l = 1, \ldots, k$. Therefore, we get

$$I_0(V) - \sum_{i=1}^{k} I_0(V_i) = -\omega_{N-1} \, C_{N,\alpha} \sum_{l=1}^{k-1} e^{-|\xi_{l+1} - \xi_l|} \left(\int_{\mathbb{R}} e^{|\rho|} W^{p_\alpha^*} \, d\rho \right) + o(\varepsilon). \tag{3.2.24}$$

Now, note that

$$I_0(V_i) = \left(\frac{2}{N-2} \right)^{\frac{2(N+\alpha)}{2+\alpha} - 1} J_0(U_{\mu_i,\alpha} + \pi_{\mu_i,\alpha}),$$

where

$$J_0(U_{\mu_i,\alpha}+\pi_{\mu_i,\alpha}) = \frac{2+\alpha}{2(N+\alpha)}\int_{\mathbb{R}^N}|x|^{\alpha}U_{1,\alpha}^{p_\alpha^*+1}\,dx + \frac{\gamma_{N,\alpha}}{2}\mu_i^{N-2}H(0,0)\int_{\mathbb{R}^N}|x|^{\alpha}U_{1,\alpha}^{p_\alpha^*}\,dx + o(\mu_i^{N-2}).$$

Finally, to estimate (3.2.3), we can proceed as

$$\begin{aligned}
\int_{\mathfrak{D}} e^{-\frac{2(\beta+2)}{N-2}\rho}V^2\,d\rho\,d\boldsymbol{\theta} &= \sum_{i=1}^{k}\int_{\mathfrak{D}_l} e^{-\frac{2(\beta+2)}{N-2}\rho}V_i^2\,d\rho\,d\boldsymbol{\theta} + o(\varepsilon)\\
&= \sum_{i=1}^{k}\int_{\mathfrak{D}_l} e^{-\frac{2(\beta+2)}{N-2}\rho}W_i^2\,d\rho\,d\boldsymbol{\theta} + o(\varepsilon)\\
&= \omega_{N-1}\sum_{i=1}^{k}\int_{\frac{\xi_{i-1}-\xi_i}{2}}^{\frac{\xi_{i+1}-\xi_i}{2}} e^{-\frac{2(\beta+2)}{N-2}(\rho+\xi_i)}W^2\,d\rho + o(\varepsilon)\\
&= \omega_{N-1}\sum_{i=1}^{k}e^{-\frac{2(\beta+2)}{N-2}\xi_i}\int_{\mathbb{R}} e^{-\frac{2(\beta+2)}{N-2}\rho}W^2\,d\rho + o(\varepsilon)\\
&= \omega_{N-1}e^{-\frac{2(\beta+2)}{N-2}\xi_1}\int_{\mathbb{R}} e^{-\frac{2(\beta+2)}{N-2}\rho}W^2\,d\rho + o(\varepsilon).
\end{aligned}$$

Thus,

$$\frac{\lambda_\varepsilon}{2}\left(\frac{2}{N-2}\right)^2\int_{\mathfrak{D}} e^{-\frac{2(\beta+2)}{N-2}\rho}V^2\,d\rho\,d\boldsymbol{\theta} = \frac{\lambda_\varepsilon}{2}\left(\frac{2}{N-2}\right)^2\omega_{N-1}e^{-\frac{2(\beta+2)}{N-2}\xi_1}\int_{\mathbb{R}} e^{-\frac{2(\beta+2)}{N-2}\rho}W^2\,d\rho + o(\varepsilon). \tag{3.2.25}$$

Then, from (3.2.20), (3.2.21), (3.2.22), (3.2.23), (3.2.24) and (3.2.25) we finishes the proof. □

3.3 The Linear Problem

In this section we consider the ansatz $v = V + \phi$ introduced before. Then we can recast problem (3.1.6) in terms of ϕ as follows:

$$\begin{cases} L_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon & \text{in } \mathfrak{D},\\ \phi = 0 & \text{on } \partial\mathfrak{D}, \end{cases} \tag{3.3.1}$$

where

$$L_\varepsilon(\phi) := L(\phi) - a_\varepsilon e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)V^{p_\alpha^*-1+\varepsilon}\phi - \lambda_\epsilon e^{-\frac{2(\beta+2)}{N-2}\rho}\left(\frac{2}{N-2}\right)^2\phi, \tag{3.3.2}$$

$$N_\varepsilon(\phi) := a_\varepsilon e^{\varepsilon\rho}((V+\phi)^{p_\alpha^*+\varepsilon} - V^{p_\alpha^*+\varepsilon} - (p_\alpha^*+\varepsilon)V^{p_\alpha^*-1+\varepsilon}\phi) \tag{3.3.3}$$

and

$$R_\varepsilon := a_\varepsilon e^{\varepsilon\rho}V^{p_\alpha^*+\varepsilon} - \sum_{i=1}^{k}W_i^{p_\alpha^*} + \lambda_\varepsilon e^{-\frac{2(\beta+2)}{N-2}\rho}\left(\frac{2}{N-2}\right)^2 V. \tag{3.3.4}$$

Before continuous, it is convenient to recall that the function

$$z_{\mu,\alpha} := \frac{\partial U_{\mu,\alpha}}{\partial \mu}$$

is the unique, up to a constant, radial solution of the equation

$$\begin{cases} -\Delta z = p_\alpha |x|^\alpha U_{\mu,\alpha}^{p_\alpha^*-1} z & \text{in } \mathbb{R}^N, \\ z \in \mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N), \end{cases} \tag{3.3.5}$$

see [21, 22] for $\alpha \geq 0$, and [2] for $-2 < \alpha < 0$.

Remark 3.3.1. *Case $\alpha = 2(m-1)$ for some $m \in \mathbb{N}$, $m \geq 2$, is more delicate because the space of solutions to problem (3.3.5) has dimension $1 + \frac{(N+2m-2)(N+m-3)!}{(N-2)!m!}$, and it is spanned by functions of the form*

$$z_{\mu,\alpha,0}(x) = z_{\mu,\alpha}(x) \quad \textit{and} \quad z_{\mu,\alpha,j}(x) = c_{\mu,m,j} \frac{Y_{m,j}(x)}{(\mu^{2+\alpha} + |x|^{2+\alpha})^{\frac{N+\alpha}{2+\alpha}}},$$

where $c_{\lambda,m,j}$ is a positive constant and $\{Y_{m,j}\}_j$, with $j = 1, 2, \dots, \frac{(N+2m-2)(N+m-3)!}{(N-2)!m!}$, is a basis of $\mathbb{Y}_m(\mathbb{R}^N)$, the space of all homogeneous harmonic polynomials of degree m in $\mathbb{R}^N$. Therefore, in both Theorem 1.4.2 and Theorem 1.4.6, the hypothesis H_2) was introduced to ensure the invertibility of the linearized operator between the appropriate function spaces.

Now, rather than solving (3.3.1) directly, we first consider the following intermediate problem: given points $\boldsymbol{\xi} := (\xi_1, \xi_2, \dots, \xi_k) \in \mathbb{R}^k$, find a function $\phi : \mathfrak{D} \to \mathbb{R}$ that is symmetric with respect to the variables $\theta_1, \theta_2, \dots, \theta_N$ and such that, for certain constants c_i, it satisfies

$$\begin{cases} L_\varepsilon(\phi) = N_\varepsilon(\phi) + R_\varepsilon + \displaystyle\sum_{i=1}^k c_i Z_i & \text{in } \mathfrak{D}, \\ \phi = 0 & \text{on } \partial\mathfrak{D}, \\ \displaystyle\int_{\mathfrak{D}} Z_i \phi \, d\rho \, d\boldsymbol{\theta} = 0 & \text{if } i = 1, 2, \dots, k, \end{cases} \tag{3.3.6}$$

where the Z_is are defined as follows. For each $i = 1, 2, \dots, k$, let $P_\Omega z_i$ be the projection onto $H^1_0(\Omega)$ of the function z_i defined by $z_i := \frac{\partial U_{\mu_i,\alpha}}{\partial \mu_i}$. We have

$$\begin{cases} \Delta P_\Omega z_i = \Delta z_i & \text{in } \Omega, \\ P_\Omega z_i = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, let $Z_i := \mathcal{T}(P_\Omega z_i)$. Then, Z_i solves

$$\begin{cases} L(Z_i) = p_\alpha^* W_i^{p_\alpha^*-1} Z_i & \text{in } \mathfrak{D}, \\ Z_i = 0 & \text{on } \partial\mathfrak{D}. \end{cases}$$

According to (3.3.6), problem (3.3.1) has been reduced to that of finding points $\boldsymbol{\xi}$ such that the constants c_is are all equal to zero. Thus, we need to solve the system

$$c_i(\boldsymbol{\xi}) = 0 \quad \text{for all } i = 1, 2, \dots, k. \tag{3.3.7}$$

If (3.3.7) holds, then $v = V + \phi$ will be a solution to (3.3.1) or equivalently to (3.1.6).

To solve problem (3.1.6), it is necessary to understand its linear part. To this end, it is convenient to introduce the following norm: given an arbitrarily small but fixed number $\sigma > 0$, we define the following weighted L^∞-norm

$$\|g\|_* := \sup_{(\rho,\boldsymbol{\theta})\in\mathfrak{D}} \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} |g(\rho,\boldsymbol{\theta})|.$$

Although this norm depends on σ and the vectors $\boldsymbol{\xi}$, we do not indicate this dependence in our notation. In fact, different choices of σ and $\boldsymbol{\xi}$ lead to equivalent norms. Additionally, we introduce the Banach space

$$\mathcal{C}^*(\mathfrak{D}) = \left\{ f \in C(\mathfrak{D}) : \|f\|_* < \infty \text{ and } f \text{ is symmetric with respect to } \theta_1, \theta_2, \dots, \theta_N \right\},$$

endowed of the $\|\cdot\|_*$-norm, and the space $\mathcal{L}(\mathcal{C}^*(\mathfrak{D}))$ of the linear operators on $\mathcal{C}^*(\mathfrak{D})$. Then, given $h \in \mathcal{C}^*(\mathfrak{D})$ such that

$$\int_{\mathfrak{D}} Z_i h \, d\rho \, d\boldsymbol{\theta} = 0 \quad \text{for } i = 1, 2, \dots, k,$$

we consider the problem of finding $\phi \in H_0^1(\mathfrak{D})$ that is symmetric with respect to the variables $\theta_1, \theta_2, \dots, \theta_N$ and such that for certain real numbers c_i, the following linear problem is satisfied:

$$\begin{cases} L_\varepsilon(\phi) = h + \sum_{i=1}^{k} c_i Z_i & \text{in } \mathfrak{D}, \\ \phi = 0 & \text{on } \mathfrak{D}, \\ \int_{\mathfrak{D}} Z_i \phi \, d\rho \, d\boldsymbol{\theta} = 0 & \text{if } i = 1, 2, \dots, k, \end{cases} \tag{3.3.8}$$

where the linear operator L_ε is defined in (3.3.2). First, we obtain the following result.

Lemma 3.3.2. *Assume the existence of sequences of numbers $\varepsilon_n \to 0$ and points $0 < \xi_1^n < \xi_2^n < \ldots < \xi_k^n$ depending on ε_n that verify*

$$\xi_1^n \to \infty, \quad \min_{i=1,\dots,k} (\xi_{i+1}^n - \xi_i^n) \to \infty \quad \text{and} \quad \xi_k^n = o(\varepsilon_n^{-1}), \tag{3.3.9}$$

such that for certain scalars $c_{i,n}$ and functions $\phi_n \in H_0^1(\mathfrak{D})$ that are symmetric with respect to the variables $\theta_1, \theta_2, \dots, \theta_N$ and $h_n \in \mathcal{C}^*(\mathfrak{D})$ with $\|h_n\|_* \to 0$ as $n \to \infty$, one has

$$\begin{cases} L_\varepsilon(\phi_n) = h_n + \sum_{i=1}^{k} c_{i,n} Z_{i,n} & \text{in } \mathfrak{D}, \\ \phi_n = 0 & \text{on } \mathfrak{D}, \\ \int_{\mathfrak{D}} Z_{i,n} \phi_n \, d\rho \, d\boldsymbol{\theta} = 0 & \text{if } i = 1, 2, \dots, k, \end{cases} \tag{3.3.10}$$

where $Z_{i,n}(\rho,\boldsymbol{\theta}) := \mathcal{T}(P_\Omega z_i)(\rho - \xi_i^n, \boldsymbol{\theta})$, for all $(\rho,\boldsymbol{\theta}) \in \mathfrak{D}$. Then, $\lim_{n\to\infty} \|\phi_n\|_* = 0$.

Proof. First, we prove that $\lim_{n\to\infty} \|\phi_n\|_\infty = 0$. Arguing by contradiction, we can assume that $\|\phi_n\|_\infty = 1$ for all $n \in \mathbb{N}$. Testing the partial differential equation in (3.3.10) with $Z_{i,n}$ and integrating by parts, we obtain

$$\int_{\mathfrak{D}} L_\varepsilon(Z_{i,n})\phi_n\, d\rho\, d\boldsymbol{\theta} - \int_{\mathfrak{D}} h_n Z_{i,n}\, d\rho\, d\boldsymbol{\theta} = \sum_{j=1}^{k} c_{i,n} \int_{\mathfrak{D}} Z_{i,n} Z_{j,n}\, d\rho\, d\boldsymbol{\theta}.$$

The previous equality defines an almost diagonal system on the $c_{i,n}$s as $n \to \infty$. Indeed,

$$\int_{\mathfrak{D}} Z_{i,n} Z_{j,n}\, d\rho\, d\boldsymbol{\theta} \to \begin{cases} 0 & \text{if } i \neq j, \\ b_i & \text{if } i = j, \end{cases}$$

as $n \to \infty$, for some values $b_i > 0$. Meanwhile, the assumption $\|h_n\|_* \to 0$ as $n \to \infty$ implies that

$$|h_n(\rho, \boldsymbol{\theta})| \leq \Theta_n(\rho, \boldsymbol{\theta}) \sum_{i=1}^{k} e^{-(1-\sigma)|\rho - \xi_i^n|}$$

for some sequence Θ_n such that $\Theta_n \to 0$ uniformly as $n \to \infty$, and bearing in mind that $Z_{i,n}(\rho, \boldsymbol{\theta}) = O\big(e^{-|\rho - \xi_i^n|}\big)$, we obtain

$$\left| \int_{\mathfrak{D}} h_n Z_{i,n}\, d\rho\, d\boldsymbol{\theta} \right| \leq C \|\Theta_n\|_\infty \left| \int_0^\infty e^{-(2-\sigma)|\rho - \xi_i^n|}\, d\rho \right| \to 0$$

as $n \to \infty$. Additionally, note that from the orthogonality condition in (3.3.10) and the fact that $Z_{i,n}(\rho, \boldsymbol{\theta})$ is a solution of $L(Z_{i,n}) = p_\alpha^* W_{i,n}^{p_\alpha^* - 1} Z_{i,n}$ in $\mathfrak{D}$, after an application of the dominated convergence theorem, it follows that

$$\int_{\mathfrak{D}} \left(a_\varepsilon (p_\alpha^* + \varepsilon_n) e^{-\varepsilon_n \rho} V^{p_\alpha^* + \varepsilon_n - 1} + \lambda_\varepsilon e^{-\frac{2(\beta+2)}{N-2}\rho} \left(\frac{2}{N-2} \right)^2 - p_\alpha^* W_{i,n}^{p_\alpha^* - 1} \right) Z_{i,n} \phi_n\, d\rho\, d\boldsymbol{\theta} \to 0$$

as $n \to \infty$. Therefore, from (3.3.10), we obtain $c_{i,n} \to 0$ as $n \to \infty$. Now, let $(\rho_n, \boldsymbol{\theta}_n) \in \mathfrak{D}$ be such that $\phi_n(\rho_n, \boldsymbol{\theta}_n) = 1$. We claim that for sufficiently large n, there exist $\bar{r} > 0$ and $i \in \{1, 2, \ldots, k\}$ such that

$$|\rho_n - \xi_i^n| < \bar{r} \qquad \text{as } n \to \infty. \tag{3.3.11}$$

Conversely, suppose that $|\rho_n - \xi_i^n| \to \infty$ as $n \to \infty$ for any $i = 1, ..., k$. Then, either $|\rho_n| \to \infty$ or $|\rho_n|$ remains bounded. First, assume that $|\rho_n| \to \infty$ as $n \to \infty$. Let us fix an index $i \in \{1, 2, \ldots, k\}$ such that (3.3.11) holds and set

$$\tilde{\phi}_n(\rho, \boldsymbol{\theta}) = \phi_n(\rho + \rho_n, \boldsymbol{\theta}), \quad (\rho, \boldsymbol{\theta}) \in \mathbb{R} \times S^{N-1}.$$

Hence, from elliptic estimates, we may assume that, up to subsequences, the sequence $\{\tilde{\phi}_n\}_{n \in \mathbb{N}}$ converges uniformly on compacts to a nontrivial solution $\tilde{\phi}$ of the equation $L(\tilde{\phi}) = 0$ in $\mathbb{R} \times S^{N-1}$ such that $\tilde{\phi} \to 0$ as $|(\rho, \boldsymbol{\theta})| \to \infty$. Hence, we can consider $\tilde{\psi}$ verifying $\tilde{\phi} = \mathcal{T}(\tilde{\psi})$ such that $\Delta \tilde{\psi} = 0$ in $\mathbb{R}^N \setminus \{0\}$. Moreover, $\|\tilde{\phi}\|_\infty = 1$ translates into $|\tilde{\psi}(y)| \leq |y|^{2-N}$. It follows that $\tilde{\psi}$ extends smoothly to 0 to a harmonic function in $\mathbb{R}^N$ with this decay condition; hence, $\tilde{\psi} = 0$, yielding a contradiction. In similar way, by assuming that $|\rho_n|$ remains bounded and putting $\tilde{\phi}_n(\rho, \boldsymbol{\theta}) = \phi_n(\rho - \rho_n + \xi_i^n, \boldsymbol{\theta})$ we get that $\tilde{\phi}_n$ converges uniformly on compacts to a function $\tilde{\phi}$, and then we get a contradiction again. Hence, we may assume that, up to subsequences, the sequence $\{\tilde{\phi}_n\}_{n\in\mathbb{N}}$ converges uniformly on compacts to a nontrivial solution $\tilde{\phi}$ of the equation

$$L(\tilde{\phi}) = p_\alpha^* W^{p_\alpha^*-1}\tilde{\phi} \quad \text{in } \mathbb{R}\times S^{N-1}. \tag{3.3.12}$$

In this way, we can consider $\tilde{\psi}$ verifying $\tilde{\phi} = \mathcal{T}(\tilde{\psi})$ such that it is bounded in $\mathbb{R}^N$ and verifies

$$-\Delta\tilde{\psi} = p_\alpha^* |x|^\alpha U_{1,\alpha}^{p_\alpha^*-1}\tilde{\psi} \quad \text{in } \ \mathbb{R}^N. \tag{3.3.13}$$

Hence, we can argue exactly as in Lemma 4.1 in [21] for obtaining $\tilde{\psi} \in \mathcal{D}^{1,2}_{\text{rad}}(\mathbb{R}^N)$ that by setting $Z := \mathcal{T}(P_\Omega z_{1,\alpha})$ implies that $\tilde{\phi} = C\,Z$ for some $C \neq 0$. However, after passing to the limit on the orthogonality condition in (3.3.10), we obtain

$$0 = \int_{\mathfrak{D}} Z_n(\rho-\xi_i^n)\phi_n(\rho)d\rho = \int_{S^{N-1}}\int_{-\xi_i^n}^{\infty} Z_n(\rho)\phi_n(\rho+\xi_i^n)d\rho \to C\int_{\mathbb{R}\times S^{N-1}} W'^2(\rho)d\rho,$$

which is a contradiction. Therefore, we have proven $\|\phi_n\|_\infty \to 0$ as $n\to\infty$.

Now, we define $g_n := L(\phi_n)$, *i.e.*,

$$g_n = a_\varepsilon e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)|V|^{p_\alpha^*-1+\varepsilon}\phi_n + h_n + \sum_{i=1}^k c_{i,n}Z_{i,n} + \lambda_\varepsilon e^{-\frac{2(\beta+2)}{N-2}\rho}\left(\frac{2}{N-2}\right)^2\phi_n. \tag{3.3.14}$$

Since $\|h_n\|_* \to 0$ as $n\to\infty$, $c_{i,n}\to 0$ as $n\to\infty$ and $L(Z_{i,n})(\rho,\boldsymbol{\theta}) = O(e^{-|\rho-\xi_i^n|})$ for $i=1,2,\dots,k$,

$$\left|a_\varepsilon(p_\alpha^*+\varepsilon)V^{p_\alpha^*-1+\varepsilon}\phi_n\right| \le C\|\phi_n\|_\infty\sum_{i=1}^k e^{-(p_\alpha^*-\sigma)|\rho-\xi_i^n|},$$

and

$$\left|\lambda_\varepsilon e^{-\frac{2(\beta+2)}{N-2}\rho}\left(\frac{2}{N-2}\right)^2\phi_n\right| \le C\|\phi_n\|_\infty\sum_{i=1}^k e^{-(1-\sigma)|\rho-\xi_i^n|},$$

with $\|\phi_n\|_\infty \to 0$ as $n\to\infty$, then

$$|g_n(\rho,\boldsymbol{\theta})| \le \Theta_n(\rho,\boldsymbol{\theta})\sum_{i=1}^k e^{-(1-\sigma)|\rho-\xi_i^n|}$$

for some $\Theta_n \to 0$ uniformly as $n\to\infty$. Choosing sufficiently large $\overline{C}>0$, we obtain that

$$\varphi_n(\rho,\boldsymbol{\theta}) = \overline{C}\,\|\Theta_n\|_\infty\sum_{i=1}^k e^{-(1-\sigma)|\rho-\xi_i^n|}$$

is a super-solution of (3.3.14) and $-\varphi_n(\rho,\boldsymbol{\theta})$ is a sub-solution of (3.3.14). Therefore,

$$|\phi_n(\rho,\boldsymbol{\theta})| \le \tilde{\Theta}_n(\rho,\boldsymbol{\theta})\sum_{i=1}^k e^{-(1-\sigma)|\rho-\xi_i^n|}$$

for some $\tilde{\Theta}_n \to 0$ uniformly as $n \to \infty$. This completes the proof. $\square$

Remark 3.3.3. *If $\alpha = 2(m-1)$ for some $m \in \mathbb{N}$, $m \in \mathbb{N}$, $m \geq 2$, and bearing in mind Remark 3.3.1, in the proof of the previous lemma, we can consider functions $\phi_n \in H_0^1(\mathfrak{D})$, with $\|\phi_n\|_*$ bounded, that are symmetric with respect to $\theta_1, \theta_2, \dots, \theta_N$ and such that ψ_n defined by $\phi_n = \mathcal{T}(\psi_n)$ in $\mathfrak{D}$ is symmetric with respect to $x_1, x_2, \dots, x_N$ and invariant with respect to the group of rotations $O(M)$, where M is an integer greater than σ_m. In this way, we can obtain $\tilde{\phi}$ that solves (3.3.12) and then $\tilde{\psi} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$ that solves (3.3.13). Since $z_{\lambda,\alpha,j}$ in Remark 3.3.1 do not belong to the family of symmetric functions with respect to $x_1, x_2, \dots, x_N$, invariant with respect to the group of rotations $O(M)$, then $\tilde{\psi} = C\, z_{1,\alpha}$ is the unique solution to (3.3.13) and $\tilde{\phi} = C\, Z$ is the unique solution to (3.3.12).*

Consider now the space

$$H_\varepsilon = \left\{ \phi \in H_0^1(\mathfrak{D}) : \int_{\mathfrak{D}} Z_i \phi \, d\rho \, d\boldsymbol{\theta} = 0, \;\; i = 1, 2, \dots, k \right\}$$

endowed with the inner product

$$(w, v)_{H_\varepsilon} = \int_{\mathfrak{D}} \phi\psi \, d\rho \, d\boldsymbol{\theta}.$$

We obtain,

Proposition 3.3.4. *There are positive numbers $\bar{\varepsilon}_k$, $\bar{\delta}_k$, $\bar{\gamma}_k$ and $\bar{r}_k$ such that if $\boldsymbol{\xi} \in \mathcal{M}_\varepsilon$, where*

$$\mathcal{M}_\varepsilon := \left\{ \boldsymbol{\xi} \in \mathbb{R}^k : \bar{r}_k < \xi_1, \quad \bar{r}_k < \min_{1 \leq i \leq k} (\xi_{i+1} - \xi_i) \quad \text{and} \quad \xi_k < \frac{\bar{\delta}_k}{\varepsilon} \right\},$$

and

$$0 < \lambda_\varepsilon < \bar{\gamma}_k,$$

then for all $\varepsilon \in (0, \bar{\varepsilon}_k)$ and all $h \in \mathcal{C}^*(\mathfrak{D})$, problem (3.3.8) admits a unique solution $\phi := T_\varepsilon(h)$. Moreover, a constant $C > 0$ exists such that $\|T_\varepsilon(h)\|_* \leq C\|h\|_*$ and $|c_i| \leq C\|h\|_*$.

We are now interested in studying the properties of the differentiability of T_ε in the variables ξ_i, which will be very important in the following. For simplicity, we will henceforth consider numbers $\bar{\varepsilon}_k$, $\bar{\delta}_k$, $\bar{\gamma}_k$ and $\bar{r}_k$, the set $\mathcal{M}_\varepsilon$ and $0 < \lambda_\varepsilon < \bar{\gamma}_k$, for $0 < \varepsilon < \bar{\varepsilon}_k$, given by Proposition 3.3.4. We define the map $S_\varepsilon : \mathcal{M}_\varepsilon \times \mathcal{C}^*(\mathfrak{D}) \to \mathcal{L}(\mathcal{C}^*(\mathfrak{D}))$ by $(\boldsymbol{\xi}, h) \mapsto S_\varepsilon(\boldsymbol{\xi}, h) = T_\varepsilon(h)$. We get,

Proposition 3.3.5. *Under the assumptions of Proposition 3.3.4, for each $h \in \mathcal{C}^*(\mathfrak{D})$, we have that the map $\boldsymbol{\xi} \mapsto S_\varepsilon(\boldsymbol{\xi}, h)$ is of class C^1. Moreover, a constant $C > 0$ exists such that $\|D_{\boldsymbol{\xi}} T_\varepsilon(h)\|_* \leq C\|h\|_*$ uniformly on vectors $\boldsymbol{\xi} \in \mathcal{M}_\varepsilon$ and values $0 < \lambda_\varepsilon < \bar{\gamma}_k$.*

For subsequent purposes, it is convenient to assume that for $M > 0$ fixed and sufficiently large, the following constraints hold:

$$\frac{1}{2}\log\frac{1}{M\varepsilon} < \xi_1, \quad \log\frac{1}{M\xi_1} < \min_{i=2,\dots,k}\{\xi_i - \xi_{i-1}\}, \quad \xi_k < k\log\frac{1}{M\varepsilon} \quad \text{and} \quad \lambda_\varepsilon < M\varepsilon^{\frac{N-4-\beta}{N-2}}. \tag{3.3.15}$$

Consider $N_\varepsilon(\phi)$ as in (3.3.3) and R_ε as in (3.3.4). We want to estimate both of them and for that, we have the following Lemmas.

Lemma 3.3.6. *Let $\xi_1, \xi_2, \dots, \xi_k$ satisfying (3.3.15). Assume additionally that $\|\phi\|_* \le \frac{1}{4}$ and σ is sufficiently small. Then, there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ we have*

$$\|N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon_0, 2\}}, \quad \|D_\phi N_\varepsilon(\phi)\|_{\mathcal{L}(\mathcal{C}^*)} \le C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon_0-1, 1\}}$$

and

$$\|\nabla_{\boldsymbol{\xi}} N_\varepsilon(\phi)\|_* \le C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon_0, 1\}}.$$

Proof. From (3.3.3), we have

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}\left((V+\phi)^{p_\alpha^*+\varepsilon} - V^{p_\alpha^*+\varepsilon} - (p_\alpha^*+\varepsilon)V^{p_\alpha^*+\varepsilon-1}\phi\right)$$

Let $f(t) = (V+t\phi)^{p_\alpha^*+\varepsilon}$. By the Mean value Theorem, there is $\nu \in (0,1)$ such that

$$(V+\phi)^{p_\alpha^*+\varepsilon} - V^{p_\alpha^*+\varepsilon} = (p_\alpha^*+\varepsilon)(V+\nu\phi)^{p_\alpha^*+\varepsilon-1}\phi$$

and this in turn implies that

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)\phi\left((V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right), \tag{3.3.16}$$

for some $\nu \in (0,1)$.
Define $g(t) = (V+t\nu\phi)^{p_\alpha^*+\varepsilon-1}$. Then, using the Mean value Theorem once again, there is $\vartheta \in (0,1)$ such that

$$(V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1} = (p_\alpha^*+\varepsilon-1)(V+\vartheta\nu\phi)^{p_\alpha^*+\varepsilon-2}\nu\phi.$$

Thus,

$$N_\varepsilon(\phi) = e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)\nu\phi^2(V+\vartheta\nu\phi)^{p_\alpha^*+\varepsilon-2}$$

To continue, we need to study 2 cases. First, assume $p_\alpha^* \ge 2$. Then, if $|V| \ge |\phi|$, we have

$$\begin{aligned} |N_\varepsilon(\phi)| \le &\ C|V+\vartheta\nu\phi|^{p_\alpha^*+\varepsilon-2}|\phi|^2 \\ \le &\ C|\phi|^2. \end{aligned}$$

On the other hand, if $|V| \le |\phi|$, then

$$\begin{aligned} |N_\varepsilon(\phi)| \le &\ C|V+\vartheta\nu\phi|^{p_\alpha^*+\varepsilon-2}|\phi|^2 \\ \le &\ C|\phi|^{p_\alpha^*+\varepsilon}. \end{aligned}$$

Onto the second case, when $p_\alpha^* < 2$, we can consider $\varepsilon > 0$ sufficiently small such that $p_\alpha^* + \varepsilon < 2$. Then, from (3.3.16), we have

$$|N_\varepsilon(\phi)| \le C|\phi|\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right|$$

Let $h(t) = (t+1)^{p_\alpha^*+\varepsilon-1} - t^{p_\alpha^*+\varepsilon-1}$. Differentiating, it is easy to see that

$$h'(t) = (p_\alpha^*+\varepsilon-1)((t+1)^{p_\alpha^*+\varepsilon-2} - t^{p_\alpha^*+\varepsilon-2}) < 0,$$

since $p_\alpha^* + \varepsilon - 2 < 0$ and this in turn implies that $h(t)$ is decreasing. Notice that $h(0) = 1$, therefore $h(t) \leq 1$ when $t \geq 0$. Thus, if $|V + \nu\phi| > |V|$, then

$$\begin{aligned}
\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right| =& \ (V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1} \\
=& \ ((V+\nu\phi)-V)^{p_\alpha^*+\varepsilon-1}\left[\left(\frac{V+\nu\phi}{(V+\nu\phi)-V}\right)^{p_\alpha^*+\varepsilon-1}\right. \\
& \left. - \left(\frac{V}{(V+\nu\phi)-V}\right)^{p_\alpha^*+\varepsilon-1}\right] \\
=& \ (\nu\phi)^{p_\alpha^*+\varepsilon-1} h\left(\frac{V}{\nu\phi}\right) \\
\leq& \ |\phi|^{p_\alpha^*+\varepsilon-1}.
\end{aligned}$$

Notice that when $|V + \nu\phi| < |V|$, then

$$\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right| = V^{p_\alpha^*+\varepsilon-1} - (V+\nu\phi)^{p_\alpha^*+\varepsilon-1}$$

and the proof is analogous to the previous case. Accordingly, $N_\varepsilon(\phi)$ can be estimate as

$$\begin{aligned}
|N_\varepsilon(\phi)| \leq& \ C\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right||\phi| \\
\leq& \ C|\phi|^{p_\alpha^*+\varepsilon-1}|\phi| \\
\leq& \ C|\phi|^{p_\alpha^*+\varepsilon}.
\end{aligned}$$

Let $\kappa(\rho) = \sup_{(\rho,\boldsymbol{\theta})\in\mathfrak{D}}\left(\sum_{i=1}^k e^{-(1-\sigma)|\rho-\xi_i|}\right)$. Thus

$$\begin{aligned}
\kappa^{-1}|N_\varepsilon(\phi)| \leq& \ C\kappa^{-1}|\phi|^{\min\{p_\alpha^*+\varepsilon,2\}} \\
\|N_\varepsilon(\phi)\|_* \leq& \ C\kappa^{\min\{p_\alpha^*+\varepsilon,2\}-1}\kappa^{-\min\{p_\alpha^*+\varepsilon,2\}}|\phi|^{\min\{p_\alpha^*+\varepsilon,2\}} \\
\leq& \ C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon,2\}}
\end{aligned}$$

For the second estimation, by direct calculus we have

$$D_\phi N_\varepsilon(\phi)\,[\psi] = e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)\left((V+\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right)\psi.$$

Following the same steps as the previous estimation, if $p_\alpha^* \geq 2$, then

$$\begin{aligned}
|D_\phi N_\varepsilon(\phi)\,[\psi]\,| =& \ e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)|V+\nu\phi|^{p_\alpha^*+\varepsilon-2}|\phi||\psi| \\
\leq& \ C|\phi|^{\min\{p_\alpha^*+\varepsilon-1,1\}}|\psi|,
\end{aligned}$$

and similaly, if $p_\alpha^* < 2$ then

$$\begin{aligned} |D_\phi N_\varepsilon(\phi)\,[\psi]\,| = \;& e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)\left|(V+\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right||\psi| \\ \leq \;& C\left|(V+\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1}\right||\psi| \\ \leq \;& C|\phi|^{p_\alpha^*+\varepsilon-1}|\psi|. \end{aligned}$$

Thus, combining both cases, we conclude that

$$\begin{aligned} |D_\phi N_\varepsilon(\phi)\,[\psi]\,| \leq \;& C|\phi|^{\min\{p_\alpha^*+\varepsilon-1,1\}}|\psi| \\ \|D_\phi N_\varepsilon(\phi)\|_{\mathcal{L}(\mathcal{C}^*)} \leq \;& C\kappa^{\min\{p_\alpha^*+\varepsilon-1,1\}}\kappa^{-\min\{p_\alpha^*+\varepsilon-1,1\}}|\phi|^{\min\{p_\alpha^*+\varepsilon-1,1\}} \\ = \;& C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon-1,1\}} \end{aligned}$$

For the last estimation, by direct calculations, we have

$$\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i} = e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)\Big((V+\phi)^{p_\alpha^*+\varepsilon-1} - V^{p_\alpha^*+\varepsilon-1} - (p_\alpha^*+\varepsilon-1)V^{p_\alpha^*+\varepsilon-2}\phi\Big)\frac{\partial V}{\partial \xi_i}.$$

Then, by using Mean value Theorem, we obtain

$$\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| = \; e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-2} - V^{p_\alpha^*+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right|.$$

If $|V| > |\phi|$, then

$$\begin{aligned} \left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| = \;& e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-2} - V^{p_\alpha^*+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ \leq \;& C|V|^{p_\alpha^*+\varepsilon-2}|\phi||V| \\ \leq \;& C|V|^{p_\alpha^*+\varepsilon-1}|\phi| \\ \leq \;& C|\phi|. \end{aligned}$$

On the other hand, if $|V| < |\phi|$, we will study two cases. First, if $p_\alpha^* \geq 2$, we can choose $\vartheta \in (0,1)$ and use the Mean Value Theorem

$$\begin{aligned} \left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| = \;& e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-2} - V^{p_\alpha^*+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ \leq \;& C|V+\vartheta\nu\phi|^{p_\alpha^*+\varepsilon-3}|\phi|^2|V|. \end{aligned}$$

If $p_\alpha^* \geq 3$, then

$$\begin{aligned} \left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| \leq \;& C|V+\vartheta\nu\phi|^{p_\alpha^*+\varepsilon-3}|\phi|^2|V| \\ \leq \;& C|\phi|^{p_\alpha^*+\varepsilon-3}|\phi|^3 \\ \leq \;& C|\phi|^{p_\alpha^*+\varepsilon} \end{aligned}$$

and if $2 \leq p_\alpha^* < 3$

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &\leq C|V+\vartheta\nu\phi|^{p_\alpha^*+\varepsilon-3}|\phi|^2|V| \\ &\leq C|V|^{p_\alpha^*+\varepsilon-3}|\phi|^2|V| \\ &\leq C|V|^{p_\alpha^*+\varepsilon-2}|\phi|^2 \\ &\leq C|\phi|^{p_\alpha^*+\varepsilon}.\end{aligned}$$

Lastly, we need to study the case $1 < p_\alpha^* < 2$. Notice that if we choose ε small enough such that $p_\alpha^* + \varepsilon - 2 < 0$, then

$$\begin{aligned}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &= e^{\varepsilon\rho}(p_\alpha^*+\varepsilon)(p_\alpha^*+\varepsilon-1)\left|(V+\nu\phi)^{p_\alpha^*+\varepsilon-2} - V^{p_\alpha^*+\varepsilon-2}\right||\phi|\left|\frac{\partial V}{\partial \xi_i}\right| \\ &\leq C|V|^{p_\alpha^*+\varepsilon-2}|\phi||V| \\ &\leq C|V|^{p_\alpha^*+\varepsilon-1}|\phi| \\ &\leq C|\phi|^{p_\alpha^*+\varepsilon}\end{aligned}$$

and this allows us to conclude that, in both cases, the following estimate holds

$$\begin{aligned}\kappa^{-1}\left|\frac{\partial N_\varepsilon(\phi)}{\partial \xi_i}\right| &\leq C\kappa^{-1}|\phi|^{\min\{p_\alpha^*+\varepsilon,1\}} \\ \|\nabla_{\boldsymbol{\xi}} N_\varepsilon(\phi)\|_* &\leq C\kappa^{\min\{p_\alpha^*+\varepsilon,1\}-1}\kappa^{-\min\{p_\alpha^*+\varepsilon,1\}}|\phi|^{\min\{p_\alpha^*+\varepsilon,1\}} \\ &\leq C\|\phi\|_*^{\min\{p_\alpha^*+\varepsilon,1\}} \quad \square.\end{aligned}$$

Lemma 3.3.7. *Let $\xi_1, \xi_2, \ldots, \xi_k$ satisfying (3.3.15). Assume additionally that σ is sufficiently small. Then*

$$\|R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}}, \quad \|\nabla_{\boldsymbol{\xi}} R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}},$$

for some $\overline{\sigma} \in (0, \frac{1}{2})$.

Proof. From (3.3.4), we can rewrite R_ε as

$$\begin{aligned}R_\varepsilon = \; & e^{\varepsilon\rho}V^{p_\alpha^*+\varepsilon} - e^{\varepsilon\rho}V^{p_\alpha^*} \\ & + e^{\varepsilon\rho}V^{p_\alpha^*} - V^{p_\alpha^*} \\ & + V^{p_\alpha^*} - \sum_{i=1}^k W_i^{p_\alpha^*} \\ & + \lambda_\varepsilon\left(\tfrac{2}{N-2}\right)^2 e^{-\frac{2(\beta+2)}{N-2}\rho}V \\ = \; & A_1 + A_2 + A_3 + A_4.\end{aligned}$$

For $A_1(\rho)$, define $f(t) = V^{p_\alpha^*+t}$. Thus, by Mean Value Theorem, we have

$$A_1(\rho) = -\varepsilon e^{\varepsilon\rho}\log(V(\rho))V^{p_\alpha^*+\nu}$$

for some $\nu \in (-\varepsilon, 0)$.

If $|V(\rho)| \geq 1$, then

$$|A_1(\rho)| \leq C\varepsilon |V(\rho)|^{p_\alpha^*+1}$$

and this in turns implies that $\|A_1\|_* \leq C\varepsilon \|V\|_* \leq C\varepsilon$.
On the other hand, if $|V(\rho)| \leq 1$, we can define $s(t) = t^{p_\alpha^*+\nu-1}\log(t)$, with $t \in [0,1]$. Note that this function reaches its maximum at $t = e^{-\frac{1}{p_\alpha^*+\nu-1}}$, so upon replacement, we obtain

$$\begin{aligned}
|A_1(\rho)| \leq\ & C\varepsilon |\log(V(\rho))|\, |V^{p_\alpha^*+\nu}| \\
\leq\ & C\varepsilon |\log(V(\rho))|\, |V^{p_\alpha^*+\nu-1}|\, |V(\rho)| \\
\leq\ & C\varepsilon \tfrac{1}{e(p_\alpha^*+\nu-1)} |V(\rho)| \\
\leq\ & C\varepsilon |V(\rho)|
\end{aligned}$$

and therefore $\|A_1\|_* \leq C\varepsilon \|V\|_* \leq C\varepsilon$.

For $A_2(\rho)$, define $g(t) = e^{t\rho}$. By Mean Value Theorem,

$$A_2(\rho) = -\varepsilon \rho e^{p\nu} V^{p_\alpha^*}$$

for some $\nu \in (-\varepsilon, 0)$. To continue, we need to analyze two cases. First, if $\rho \leq \xi_k$, then

$$\begin{aligned}
\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \geq\ & e^{-(1-\sigma)|\rho-\xi_k|} \\
=\ & e^{-(1-\sigma)(\xi_k-\rho)}
\end{aligned}$$

and such

$$\left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}\right)^{-1} \leq e^{(1-\sigma)(\xi_k-\rho)}.$$

Even more, from (3.3.15), we have $e^{(1-\sigma)\xi_k} \leq (M\varepsilon)^{-(1-\sigma)k}$. Thus

$$\begin{aligned}
\left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}\right)^{-1} |A_2(\rho)| \leq\ & \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}\right)^{-1} \varepsilon\rho |V|^{p_\alpha^*} \\
\leq\ & e^{(1-\sigma)(\xi_k-\rho)} \varepsilon\rho |V|^{p_\alpha^*} \\
\leq\ & \varepsilon \|V\|_\infty^{p_\alpha^*} (\rho e^{-(1-\sigma)\rho}) e^{(1-\sigma)\xi_k} \\
\leq\ & C\varepsilon (M\varepsilon)^{-(1-\sigma)k} \\
\leq\ & C\varepsilon^{1-(1-\sigma)k}.
\end{aligned}$$

Conversely, if $\rho > \xi_k$, then

$$\begin{aligned}
\frac{|V(\rho)|^{p_\alpha^*}}{\sum\limits_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}} \leq\ & \frac{Ce^{-p_\alpha^*|\rho-\xi_k|}}{ke^{-(1-\sigma)|\rho-\xi_k|}} \\
=\ & Ce^{-(p_\alpha^*-1+\sigma)(\rho-\xi_k)}
\end{aligned}$$

and such

$$\begin{aligned}\left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}\right)^{-1}|A_2(\rho)| \leq & \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|}\right)^{-1}\varepsilon\rho|V|^{p^*_\alpha}\\ \leq & \ C\varepsilon\rho e^{-(p^*_\alpha-1+\sigma)(\rho-\xi_k)}\\ \leq & \ C\varepsilon\rho e^{-\sigma(\rho-\xi_k)}\\ = & \ C\varepsilon(\rho e^{-\sigma\rho})e^{\sigma\xi_k}\\ \leq & \ C\varepsilon^{1-k\sigma},\end{aligned}$$

for σ small enough. Therefore, $\|A_2\|_* \leq C\varepsilon^{1-k\sigma}$.

Next, to estimate $A_3(\rho)$, we need to introduce a positive constant $\omega \in (0,1)$, which will be determine later. Assume that there is $l \in \{1,\dots,k\}$ such that $|\rho-\xi_l| < \omega\log(\frac{1}{M\varepsilon})$. From (2.3.13), we have

$$\begin{aligned}|\rho-\xi_i| > & \ |\xi_l-\xi_i|-|\rho-\xi_l|\\ > & \ \log(\tfrac{1}{M\varepsilon})-\omega\log(\tfrac{1}{M\varepsilon})\\ = & \ (1-\omega)\log(\tfrac{1}{M\varepsilon})\end{aligned}$$

and this implies that $e^{-|\rho-\xi_i|} \leq M\varepsilon^{(1-\omega)}$, for $i \neq l$. Even more, the previous inequality implies that $|W_i| \leq M\varepsilon^{(1-\omega)}, |\Pi_i| \leq M\varepsilon^{(1-\omega)}$ and $|V_i| \leq M\varepsilon^{(1-\omega)}$ if $i \neq l$. Moreover, we notice that $|\rho-\xi_l| < \omega\log(\frac{1}{M\varepsilon})$ also indicates

$$\begin{aligned}-\omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho-\xi_l\\ 2\xi_l-\omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho+\xi_l\\ 2\xi_1-\omega\log(\tfrac{1}{M\varepsilon}) < & \ \rho+\xi_l,\end{aligned}$$

thus

$$\begin{aligned}|\Pi_l(\rho)| \leq & \ Ce^{-(\rho+\xi_l)}\\ \leq & \ Ce^{-2\xi_1+\omega\log(\frac{1}{M\varepsilon})}\\ \leq & \ Ce^{-2\xi_1}\varepsilon^{-\omega}\\ \leq & \ C\varepsilon^{1-\omega}.\end{aligned}$$

Having stated that, define

$$h(t) = \left(W_l(\rho)+t\left(\Pi_l(\rho)+\sum_{i=1,i\neq l}^{k} V_i(\rho)\right)\right)^{p^*_\alpha},$$

then, by Mean Value Theorem, we have

$$V^{p^*_\alpha}(\rho)-W_l^{p^*_\alpha}(\rho) = p^*_\alpha\left(W_l(\rho)+\nu\left(\Pi_l(\rho)+\sum_{i=1,i\neq l}^{k} V_i(\rho)\right)\right)^{p^*_\alpha-1}\left(\Pi_l(\rho)+\sum_{i=1,i\neq l}^{k} V_i(\rho)\right)$$

for some $\nu \in (0,1)$. Hence

$$\begin{aligned}
\|A_3(\rho)\|_* = \kappa^{-1} & \left| p_\alpha^* \left(W_l(\rho) + \nu \left(\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho) \right) \right)^{p_\alpha^*-1} \left(\Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho) \right) \right. \\
& \left. - \sum_{i=1, i\neq l}^{k} W_i^{p_\alpha^*}(\rho) \right| \\
\leq \ & C \left| \Pi_l(\rho) + \sum_{i=1, i\neq l}^{k} V_i(\rho) \right| + \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} \left| \sum_{i=1, i\neq l}^{k} W_i^{p_\alpha^*}(\rho) \right| \\
\leq \ & C\varepsilon^{(1-\omega)} + C \sum_{i=1, i\neq l}^{k} e^{-(p_\alpha^*-1+\sigma)|\rho-\xi_i|} \\
\leq \ & C\varepsilon^{(1-\omega)} + C\varepsilon^{(p_\alpha^*-1+\sigma)(1-\omega)} \\
\leq \ & C\varepsilon^{(1-\omega)}
\end{aligned}$$

Now assume the opposite, in other words, let $|\rho - \xi_i| \geq \omega \log(\frac{1}{M\varepsilon})$ for all $i \in \{1, \ldots, k\}$. It is easy to see that this implies $e^{-|\rho-\xi_i|} \leq M\varepsilon^\omega$ for all $i \in \{1, \ldots, k\}$. This in turn implies that

$$\begin{aligned}
V_i(\rho) \leq \ & Ce^{-|\rho-\xi_i|} \\
\leq \ & C\varepsilon^\omega
\end{aligned}$$

for all $i \in \{1, \ldots, k\}$ and the same applies to $W_i(\rho)$ for all $i \in \{1, \ldots, k\}$. Thus, we can estimate

$$\begin{aligned}
\|A_3\|_* \leq \ & \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} |V(\rho)|^{p_\alpha^*} + \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} \sum_{i=1}^{k} |W_i(\rho)|^{p_\alpha^*} \\
\leq \ & C\varepsilon^{-(1-\sigma)\omega} \varepsilon^{p_\alpha^*\omega} + C\varepsilon^{-(1-\sigma)\omega} \varepsilon^{p_\alpha^*\omega} \\
= \ & C\varepsilon^{(p_\alpha^*-1+\sigma)\omega}.
\end{aligned}$$

Choosing ω such that $(p_\alpha^* - 1 + \sigma)\omega = 1 - \omega$, in other words $\omega = \frac{1}{p_\alpha^*+\sigma}$, we can conclude that $\|A_3\|_* \leq C\varepsilon^{(1-\omega)}$.

Lastly, for $A_4(\rho)$, we have

$$\begin{aligned}
\left(\sum_{i=1}^{k} e^{-\sigma|\rho-\xi_i|} \right)^{-1} |A_4(\rho)| & \leq \eta\varepsilon^{\frac{N-4-\beta}{N-2}} \left(\frac{2}{N-2} \right)^2 e^{-\frac{2(\beta+2)}{N-2}\rho} \sum_{i=1}^{k} e^{-|\rho-\xi_i|} \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} \\
& \leq \eta\varepsilon^{\frac{N-4-\beta}{N-2}} \left(\frac{2}{N-2} \right)^2 e^{-\frac{2(\beta+2)}{N-2}\rho} \sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \left(\sum_{i=1}^{k} e^{-(1-\sigma)|\rho-\xi_i|} \right)^{-1} \\
& \leq C\varepsilon^{\frac{N-4-\beta}{N-2}}
\end{aligned}$$

thus $\|A_4\|_* \leq C\varepsilon^{\frac{N-4-\beta}{N-2}}$.
Therefore, $\|R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}}$, where $\overline{\sigma} = \min\{(1-k\sigma), 1 - \frac{1}{p_\alpha^*+\sigma}, \frac{N-4-\beta}{N-2}\}$.

To prove $\|\nabla_{\boldsymbol{\xi}} R_\varepsilon\|_* \leq C\varepsilon^{\overline{\sigma}}$, notice that

$$\frac{\partial R_\varepsilon}{\partial \xi_i} = \frac{\partial A_1}{\partial \xi_i} + \frac{\partial A_2}{\partial \xi_i} + \frac{\partial A_3}{\partial \xi_i} + \frac{\partial A_4}{\partial \xi_i},$$

for all $i = \{1, \dots, k\}$. Then, we can use the previous procedure to conclude the proof. $\square$

Thus, we obtain the next result,

Proposition 3.3.8. *Assume that constraints (3.3.15) hold. Then, $C > 0$ exists such that for all sufficiently small ε, a unique solution $\phi = \phi(\boldsymbol{\xi})$ to problem (3.3.6) exists. Moreover, the map $\boldsymbol{\xi} \mapsto \phi(\boldsymbol{\xi})$ is of C^1-class for the $\|\cdot\|_*$-norm and satisfies $\|\phi\|_* \leq C\varepsilon^{\overline{\sigma}}$ and $\|D_{\boldsymbol{\xi}}\phi\|_* \leq C\varepsilon^{\overline{\sigma}}$, for some $\overline{\sigma} \in (\frac{1}{2}, 1)$.*

3.4 Finite dimensional reduction

Finally, we consider the constraints (3.3.15) and the function $\phi = \phi(\boldsymbol{\xi})$ given by Proposition 3.3.8, which is the only solution of problem (3.3.6). According to previous calculations, we note that $c_i = 0$ in (3.3.6), for all $i = 1, 2, \dots, k$, is equivalent to saying that $v = V + \phi(\boldsymbol{\xi})$ is a solution of (3.1.6). Therefore, we need to find points $\boldsymbol{\xi}$ so that the system $c_i(\boldsymbol{\xi}) = 0$ for all $i = 1, 2, \dots, k$ has solution. This system turns out to be equivalent to a variational problem, as stated below. Consider

$$\mathfrak{E}_\varepsilon(\boldsymbol{\xi}) := E_\varepsilon(V + \phi(\boldsymbol{\xi})), \tag{3.4.1}$$

where E_ε is the functional given in (3.2.1). Hence, by means of standard arguments we get,

Lemma 3.4.1. *Assume that constraints (3.3.15) hold and let $\phi = \phi(\boldsymbol{\xi})$ the function given by Proposition 3.3.8 and V as in (3.2.5). Then, the function $v = V + \phi(\boldsymbol{\xi})$ is a solution of (3.1.6) if and only if $\boldsymbol{\xi}$ is a critical point of $\mathfrak{E}_\varepsilon$.*

The next step is to validate an expansion for $\mathfrak{E}_\varepsilon$, which will be crucial for finding its critical points.

Proposition 3.4.2. *Assume that constraints (3.3.15) hold, and let $\phi = \phi(\boldsymbol{\xi})$ the function given by Proposition 3.3.8 and V as in (3.2.5). Under the assumptions of Lemma 3.2.2, the following expansion holds:*

$$\mathfrak{E}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V) + o(\varepsilon),$$

where $o(\varepsilon)$ is uniformly of this size in the C^1-sense on the vectors $\boldsymbol{\xi}$.

3.5 Proof of Theorem 1.4.2

Proof of Theorem 1.4.2. Consider the change of variables (3.2.6). Thus, from (3.4.1) and Lemma 3.4.1, it suffices to find a critical point of the functional $\Phi_\varepsilon : (0, \infty)^k \to \mathbb{R}$ defined by

$$\Phi_\varepsilon(\boldsymbol{\mu}) := \varepsilon^{-1}\mathfrak{E}_\varepsilon(\boldsymbol{\xi}(\boldsymbol{\mu})).$$

From Proposition 3.4.2 and the expansion given by Lemma 3.2.2, we obtain

$$\frac{\partial \Phi_\varepsilon}{\partial \mu_i}(\boldsymbol{\mu}) = \frac{\partial \Psi_k}{\partial \mu_i}(\boldsymbol{\mu}) + \boldsymbol{o}(1) \quad \text{for all } i = 1, 2, \ldots, k,$$

where $\boldsymbol{o}(1) \to 0$ uniformly on the vectors $\boldsymbol{\mu}$ satisfying $M^{-1} < \mu_i < M$ for any fixed positive M that is sufficiently large, and Ψ_k is the functional defined in (3.2.8). We claim that there exist $\eta_k > 0$, such that for $\eta > \eta_k$, the function Ψ_k has exactly two nondegenerate critical points: $\boldsymbol{\mu}^{+*} = (\mu_1^{+*}, \mu_2^*, \ldots, \mu_k^*)$ and $\boldsymbol{\mu}^{-*} = (\mu_1^{-*}, \mu_2^*, \ldots, \mu_k^*)$, where μ_1^{+*} and μ_1^{-*} are respectively a strictly maximum and strictly minimum on $(0, \infty)$ of the function

$$\mu_1 \mapsto a_5 \mu_1^{-2} - a_2 k \log \mu_1 - a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}}$$

and μ_i^* is a strictly maximum on $(0, \infty)$ of the function

$$\mu_i \mapsto (k - i + 1) a_2 \log \mu_i - a_3 \mu_i \ \text{ for } i = 2, 3, \ldots, k,$$

where constants a_is are given in (3.2.9)-(3.2.14). Indeed, let

$$f(\mu_1) = a_5 \mu_1^{-2} - a_2 k \log \mu_1 - a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}}.$$

By differentiating and equating to zero, we get

$$f'(\mu_1) = -2a_5 \mu_1^{-3} - k a_2 \mu_1^{-1} + 2\frac{\beta+2}{N-2} a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}-1} = 0,$$

from where it follows that

$$\eta = \frac{a_5}{a_6} \frac{N-2}{\beta+2} \mu_1^{2\frac{\beta+2}{N-2}-2} + \frac{k}{2} \frac{a_2}{a_6} \frac{N-2}{\beta+2} \mu_1^{2\frac{\beta+2}{N-2}} > 0.$$

Consider now the function

$$g(s) = \frac{a_5}{a_6} \frac{N-2}{\beta+2} s^{2\frac{\beta+2}{N-2}-2} + \frac{k}{2} \frac{a_2}{a_6} \frac{N-2}{\beta+2} s^{2\frac{\beta+2}{N-2}}$$

which reaches a unique strict positive minimum at $s = s^* > 0$. This leads to that f possesses exactly two critical points: a strictly minimum point μ_1^{-*} and a strictly maximum point μ_1^{+*} that verify $0 < \mu_1^{-*} < \mu_1^{+*}$ and $f''(\mu_1^{-*}) > 0$ and $f''(\mu_1^{+*}) < 0$ for $\eta > \eta_k$, with η_k chosen sufficiently large. On the other hand, for each $i = 2, 3, \ldots, k$, the function

$$h_i(\mu_i) = a_2 (k - i + 1) \log \mu_i - a_3 \mu_i$$

verifies $h_i'(\mu_i) = a_2(k - i + 1)\mu_i^{-1} - a_3$ and $h_i''(\mu_i) = -a_2(k - i + 1)\mu_i^{-2}$, and has a unique strict maximum point at $\mu_i^* = \frac{a_2(k-i+1)}{a_3}$. Consequently, $\boldsymbol{\mu}^{\pm *} = \left(\mu_1^{\pm *}, \mu_2^*, \ldots, \mu_k^*\right)$ are the only critical points of the functional Ψ_k, which are non-degenerate.

It follows that $\nabla \Psi_k$ is stable with respect to small and uniform perturbations. Hence, if $\mathfrak{V}^{\pm}$ are arbitrary neighborhoods of $\boldsymbol{\mu}^{\pm *}$ in $(0, \infty)^k$ with quite small diameters and such that $\overline{\mathfrak{V}}^{\pm} \subset$

$(0,\infty)^k$, then the topological degrees $\deg(\nabla\Psi_k, \mathfrak{V}^{\pm}, 0)$ are well defined and $\deg(\nabla\Psi_k, \mathfrak{V}^{\pm}, 0) \neq 0$. By considering the homotopy $\mathcal{H}_t = t\nabla\Phi_\varepsilon + (1-t)\nabla\Psi_k$, on $\mathfrak{V}^{\pm}$, for $0 \leq t \leq 1$, we get

$$\deg(\nabla\Phi_\varepsilon, \mathfrak{V}^{\pm}, 0) = \deg(\nabla\Psi_k, \mathfrak{V}^{\pm}, 0) \neq 0$$

for all $\varepsilon > 0$ sufficiently small. Thus, for $\varepsilon > 0$ small enough, there exists a critical point $\boldsymbol{\mu}^{\pm}_{\varepsilon} = (\mu^{\pm}_{1,\varepsilon}, \mu_{2,\varepsilon}, \ldots, \mu_{k,\varepsilon})$ of Φ_ε such that

$$\mu^{\pm}_{1,\varepsilon} = \mu^{\pm *}_1 + o(1) \quad \text{and} \quad \mu_{i,\varepsilon} = \mu^*_i + o(1) \quad \text{for all} \;\; i = 2, 3, \ldots, k,$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Furthermore, due to change of variables (3.6.9), this is equivalent to saying that the points $\boldsymbol{\xi}^{\pm}_{\varepsilon} = (\xi^{\pm}_{1,\varepsilon}, \xi^{\pm}_{2,\varepsilon}, \ldots, \xi^{\pm}_{k,\varepsilon})$, where

$$\xi^{\pm}_{1,\varepsilon} = \log\frac{\mu^{\pm}_{1,\varepsilon}}{\sqrt{\varepsilon}} \quad \text{and} \quad \xi^{\pm}_{i,\varepsilon} = \log\frac{\mu^{\pm}_{1,\varepsilon}}{\mu_{2,\varepsilon}\cdots\mu_{i,\varepsilon}\varepsilon^{\frac{2i-1}{2}}} \;\; \text{if } i = 2, 3, \ldots, k,$$

are critical points of $\mathfrak{E}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V + \phi(\boldsymbol{\xi}))$ and, from Lemma 3.4.1, functions $v^{\pm}_{k,\varepsilon} = V + \phi(\boldsymbol{\xi}^{\pm}_{\varepsilon})$ are solutions to (3.1.6). Therefore,

$$v^{\pm}_{k,\varepsilon}(\rho,\boldsymbol{\theta}) = \sum_{i=1}^{k}\left(W\left(\rho - \xi^{\pm}_{i,\varepsilon}\right) + \Pi_{\xi^{\pm}_{i,\varepsilon},\alpha}(\rho,\boldsymbol{\theta})\right) + \phi\left(\boldsymbol{\xi}^{\pm}_{\varepsilon}\right)(\rho,\boldsymbol{\theta}) \quad \text{for all } (\rho,\boldsymbol{\theta}) \in \mathfrak{D},$$

is a solution of (3.1.6), and thus, going back in the change of variables, the function

$$u^{\pm}_{k,\varepsilon}(x) = \gamma_{N,\alpha}\sum_{i=1}^{k}\left(\left(\frac{e^{-\frac{2+\alpha}{N-2}\xi^{\pm}_{i,\varepsilon}}}{e^{-\frac{2(2+\alpha)}{N-2}\xi^{\pm}_{i,\varepsilon}} + |x|^{2+\alpha}}\right)^{\frac{N-2}{2+\alpha}} + \pi_{\mu^{\pm}_{i,\varepsilon},\alpha}(x)\right) + \varphi_\varepsilon(x) \quad \text{for all } x \in \Omega,$$

where $\varphi_\varepsilon(x) \to 0$ uniformly on compacts contained in Ω as $\varepsilon \to 0$ and $\mu^{\pm}_{i,\varepsilon} = e^{-\frac{2}{N-2}\xi^{\pm}_{i,\varepsilon}} = M_i\varepsilon^{\frac{2i-1}{N-2}}$, is a solution of (3.1.1). This finishes the proof. □

3.6 Proof of Theorem 1.4.6

Consider again spherical coordinates $x = x(\rho,\boldsymbol{\theta})$ centered at the origin, where ρ and $\boldsymbol{\theta}$ are given by $\rho = |x|$ and $\boldsymbol{\theta} = \frac{x}{|x|}$, and the transformation $\mathcal{T}$

$$v(\rho,\boldsymbol{\theta}) := \mathcal{T}(u)(\rho,\boldsymbol{\theta}) = \left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}} e^{\rho}u(e^{\frac{2}{N-2}\rho}\boldsymbol{\theta}), \qquad \text{for all } (\rho,\boldsymbol{\theta}) \in \mathfrak{D}. \tag{3.6.1}$$

where $\mathfrak{D} := \{(\rho, \boldsymbol{\theta}) : \rho > \kappa_0(\boldsymbol{\theta}),\ \boldsymbol{\theta} \in S^{N-1}\}$, with $\kappa_0 : S^{N-1} \to (0, \infty)$ is a continuous function. After this change of variables, problem (1.4.5) becomes

$$\begin{cases} L(v) = \tilde{a}_\varepsilon e^{\rho\varepsilon}|v|^{p_\alpha^* - \varepsilon} - \lambda_\varepsilon \left(\dfrac{2}{N-2}\right)^2 e^{\frac{2(\beta+2)}{N-2}\rho} v & \text{in } \mathfrak{D}, \\ v = 0 & \text{on } \partial\mathfrak{D}, \\ \lim\limits_{|\rho| \to \infty} v(\rho, \boldsymbol{\theta}) = 0, \end{cases} \tag{3.6.2}$$

where $\tilde{a}_\varepsilon := \left(\frac{N-2}{2}\right)^{-\frac{N-2}{2+\alpha}\varepsilon}$ and L_ε is as in (3.1.7).

The energy functional to study now is

$$E_\varepsilon(v) := I_\varepsilon(v) + K_\varepsilon(v), \tag{3.6.3}$$

where

$$I_\varepsilon(v) := \frac{1}{2}\int_{\mathfrak{D}} \left(\left(\frac{2}{N-2}\right)^2 |\nabla_\theta v|^2 + \left|\frac{\partial v}{\partial \rho}\right|^2 + |v|^2\right) d\rho\, d\boldsymbol{\theta} - \frac{\tilde{a}_\varepsilon}{p_\alpha^* + 1 - \varepsilon}\int_{\mathfrak{D}} e^{\rho\varepsilon}|v|^{p_\alpha^* + 1 - \varepsilon}\, d\rho\, d\boldsymbol{\theta} \tag{3.6.4}$$

and

$$K_\varepsilon(v) := \frac{\lambda_\varepsilon}{2}\left(\frac{2}{N-2}\right)^2 \int_{\mathfrak{D}} e^{\frac{2(\beta+2)}{N-2}\rho}|v|^2\, d\rho\, d\boldsymbol{\theta}. \tag{3.6.5}$$

Similarly as before, for $i = 1, \dots, k$, let $\xi_i, \mu_i \in \mathbb{R}$ be values such that $0 < \xi_1 < \cdots < \xi_k$ and $\xi_i = \frac{N-2}{2}\log \mu_i$. We seek a solution of (3.6.2) of the form

$$v_{k,\varepsilon}(\rho, \boldsymbol{\theta}) = \sum_{i=1}^{k} \left(W(\rho - \xi_i) + \Pi_{\xi_i,\alpha}(\rho, \boldsymbol{\theta})\right) + \phi(\rho, \boldsymbol{\theta}), \quad (\rho, \boldsymbol{\theta}) \in \mathfrak{D}, \tag{3.6.6}$$

where $\Pi_{\xi,\alpha}$ solves the problem

$$\begin{cases} L(\Pi_{\xi,\alpha}) = 0 & \text{in } \mathfrak{D}, \\ \Pi_{\xi,\alpha} = -W(\cdot - \xi) & \text{on } \partial\mathfrak{D}. \end{cases} \tag{3.6.7}$$

All results in Section 3 are valid in this setting. Now, it is convenient to introduce the notation

$$W_i(\rho) := W(\rho - \xi_i), \quad \Pi_i := \Pi_{\xi_i,\alpha}, \quad V_i := W_i + \Pi_i, \quad V := \sum_{i=0}^{k} V_i, \tag{3.6.8}$$

and choose appropriate parameters ξ_is depending on μ_is, namely

$$\xi_1 = -\frac{1}{2}\log\varepsilon - \log\mu_1, \quad \xi_{i+1} - \xi_i = -\log\varepsilon + \log\mu_{i+1}, \quad \text{for all } i = 1, 2, \dots, k-1. \tag{3.6.9}$$

where μ_is are positive parameters to be determined. We get

Lemma 3.6.1. *Let $k \in \mathbb{N}, k \geq 2$ and $\delta > 0$. Asssume that*

$$\delta < \mu_i < \delta^{-1} \quad \textit{for } i = 1, 2, \ldots, k. \tag{3.6.10}$$

Futhermore, consider $\lambda_\varepsilon = \eta \varepsilon^{\frac{N+\beta}{N-2}}$, for some $\eta > 0$ sufficiently large. Then, for V defined in (3.6.8) and points $\xi_i s$ in (3.6.9), there are positive numbers a_1, a_2, a_3, a_4, a_5 y a_6 depending only on N and α such that

$$E_\varepsilon(V) = k a_1 + \varepsilon \Psi_k(\boldsymbol{\mu}) + \frac{k^2}{2}(\varepsilon \log \varepsilon) a_2 + k \varepsilon a_4 + \varepsilon \Theta_\varepsilon(\boldsymbol{\mu}),$$

where $\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_k)$,

$$\Psi_k(\boldsymbol{\mu}) = -a_3 \sum_{i=2}^{k} \frac{1}{\mu_i} - a_2 \sum_{i=2}^{k} (k - i + 1) \log \mu_i + k a_2 \log \mu_1 + a_5 \mu_1^2 + a_6 \eta \mu_1^{-2\frac{(\beta+2)}{N-2}} \tag{3.6.11}$$

and $\Theta_\varepsilon(\boldsymbol{\mu}) \to 0$ as $\varepsilon \to 0$ uniformly with respect to the values of μ_i satisfying (3.6.10), where a_1, a_2, a_3 and a_6 are given by (3.2.9), (3.2.10), (3.2.11) and (3.2.14) respectively, and

$$a_4 := -\frac{\omega_{N-1}}{p_\alpha^* + 1}\left(\left(\frac{1}{p_\alpha^* + 1} - \log\left(\frac{N-2}{2}\right)^{\frac{N-2}{2+\alpha}}\right) \int_{-\infty}^{\infty} W^{p_\alpha^*+1} d\rho - \int_{-\infty}^{\infty} W^{p_\alpha^*+1} \log W \, d\rho\right) \tag{3.6.12}$$

and

$$a_5 := \left(\frac{4(N+\alpha)}{N-2}\right)^{\frac{N-2}{2+\alpha}}. \tag{3.6.13}$$

The proof of Lemma 3.6.1 is very similar to the proof of Lemma 3.2.2. The main difference is to avoid using the regular part of Green's function, which is achieved by a Taylor expansion of $I_0(V_i)$.

Proof of Theorem 1.4.6. We consider the change of variables (3.6.9). Thus, from (3.4.1) and Lemma 3.4.1, it suffices to find a critical point of the functional $\Phi_\varepsilon : (0, \infty)^k \to \mathbb{R}$ defined by

$$\Phi_\varepsilon(\boldsymbol{\mu}) := \varepsilon^{-1} \mathfrak{E}_\varepsilon(\boldsymbol{\xi}(\boldsymbol{\mu})).$$

From Proposition 3.4.2 and the expansion given by Lemma 3.2.2, we obtain

$$\frac{\partial \Phi_\varepsilon}{\partial \mu_i}(\boldsymbol{\mu}) = \frac{\partial \Psi_k}{\partial \mu_i}(\boldsymbol{\mu}) + \boldsymbol{o}(1) \quad \text{for all } i = 1, 2, \ldots, k,$$

where $\boldsymbol{o}(1) \to 0$ uniformly on the vectors $\boldsymbol{\mu}$ satisfying $\delta^{-1} < \mu_i < \delta$ for any fixed positive δ that is sufficiently large, and Ψ_k is the functional defined in (3.6.11). We claim that Ψ_k has one non degenerate critical point $\boldsymbol{\mu}^* = (\mu_1^*, \mu_2^*, \ldots, \mu_k^*)$, where μ_1^* and μ_i^* are, respectively, critical points in $]0, +\infty[$ of the functions

$$\mu_1 \mapsto a_5 \mu_1^2 + a_2 k \log \mu_1 + a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}} \quad \text{and} \quad \mu_i \mapsto -a_2(k - i + 1) \log \mu_i - a_3 \mu_i^{-1},$$

if $i = 2, 3, \ldots, k$ and the constants a_j are given by (3.2.9), (3.2.10), (3.2.11), (3.6.12), (3.6.13) and

(3.2.14) respectively. Indeed, let

$$f(\mu_1) = a_5\mu_1^2 + a_2 k \log \mu_1 + a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}}.$$

By differentiating and equating to zero, we get

$$f'(\mu_1) = 2a_5\mu_1 + ka_2\mu_1^{-1} - 2\frac{\beta+2}{N-2} a_6 \eta \mu_1^{-2\frac{\beta+2}{N-2}-1} = 0,$$

from where it follows that

$$\eta = \frac{a_5}{a_6}\frac{N-2}{\beta+2}\mu_1^{2\frac{\beta+2}{N-2}+2} + \frac{k}{2}\frac{a_2}{a_6}\frac{N-2}{\beta+2}\mu_1^{2\frac{\beta+2}{N-2}} > 0.$$

Consider now the function

$$g(s) = \frac{a_5}{a_6}\frac{N-2}{\beta+2}s^{2\frac{\beta+2}{N-2}+2} + \frac{k}{2}\frac{a_2}{a_6}\frac{N-2}{\beta+2}s^{2\frac{\beta+2}{N-2}}$$

which reaches a unique strict minimum at $s = 0$. This leads to that f possesses a unique strict minimum point μ_1^* which verifies $f''(\mu_1^*) > 0$ for $\eta > \eta_k$, with η_k chosen sufficiently large. On the other hand, for each $i = 2, 3, \ldots, k$, the function

$$h_i(\mu_i) = -a_2(k-i+1)\log \mu_i - a_3\mu_i^{-1}$$

verifies $h_i'(\mu_i) = -a_2(k-i+1)\mu_i^{-1} + a_3\mu_i^{-2}$ and $h_i''(\mu_i) = a_2(k-i+1)\mu_i^{-2} - 2a_3\mu_i^{-3}$, and has a unique strict maximum point at $\mu_i^* = \frac{a_3}{a_2(k-i+1)}$. Consequently, $\boldsymbol{\mu}^* = (\mu_1^*, \mu_2^*, \ldots, \mu_k^*)$ is the only one critical point of the functional Ψ_k, which is non degenerate.

It follows that $\nabla\Psi_k$ is stable with respect to small and uniform perturbations. Hence, if $\mathfrak{V}$ is an arbitrary neighborhood of $\boldsymbol{\mu}^*$ in $(0,\infty)^k$ with quite small diameter such that $\overline{\mathfrak{V}} \subset (0,\infty)^k$, then the topological degree $\deg(\nabla\Psi_k, \mathfrak{V}, 0)$ is well defined and $\deg(\nabla\Psi_k, \mathfrak{V}, 0) \neq 0$. By considering the homotopy $\mathcal{H}_t = t\nabla\Phi_\varepsilon + (1-t)\nabla\Psi_k$, on $\mathfrak{V}$, for $0 \le t \le 1$, we get

$$\deg(\nabla\Phi_\varepsilon, \mathfrak{V}, 0) = \deg(\nabla\Psi_k, \mathfrak{V}, 0) \neq 0$$

for all $\varepsilon > 0$ sufficiently small. Thus, for $\varepsilon > 0$ small enough, there exists a critical point $\boldsymbol{\mu}_\varepsilon^* = (\mu_{1,\varepsilon}^*, \mu_{2,\varepsilon}^*, \ldots, \mu_{k,\varepsilon}^*)$ of Φ_ε such that

$$\mu_{1,\varepsilon} = \mu_1^* + o(1) \quad \text{and} \quad \mu_{i,\varepsilon} = \mu_i^* + o(1) \quad \text{for all } \; i = 2, 3, \ldots, k,$$

where $o(1) \to 0$ as $\varepsilon \to 0$. Furthermore, due to change of variables (3.6.9), this is equivalent to saying that the point $\boldsymbol{\xi}_\varepsilon = (\xi_{1,\varepsilon}, \xi_{2,\varepsilon}, \ldots, \xi_{k,\varepsilon})$, where

$$\xi_{1,\varepsilon} = \log \frac{1}{\sqrt{\varepsilon}\mu_{1,\varepsilon}} \qquad \text{and} \qquad \xi_{i,\varepsilon} = \log \frac{\mu_{2,\varepsilon}\cdots\mu_{i,\varepsilon}}{\mu_{1,\varepsilon}\varepsilon^{\frac{2i-1}{2}}} \quad \text{for all } \; i = 2, 3, \ldots, k,$$

is the critical point of $\mathfrak{E}_\varepsilon(\boldsymbol{\xi}) = E_\varepsilon(V + \phi(\boldsymbol{\xi}))$ and, from Lemma 3.4.1, the function $v_{k,\varepsilon} = V + \phi(\boldsymbol{\xi}_\varepsilon)$

is a solution to (3.6.2). Therefore,

$$v_{k,\varepsilon}(\rho,\boldsymbol{\theta}) = \sum_{i=1}^{k} \big(W(\rho-\xi_{i,\varepsilon}) + \Pi_{\xi_{i,\varepsilon},\alpha}(\rho,\boldsymbol{\theta})\big) + \phi(\boldsymbol{\xi}_\varepsilon)(\rho,\boldsymbol{\theta}),$$

for all $(\rho,\boldsymbol{\theta}) \in \mathfrak{D}$ is the solution of (3.6.2). Then, going back to the original variables, we conclude that

$$u_{k,\varepsilon}(x) = \gamma_{N,\alpha} \sum_{i=1}^{k} \left(\left(\frac{e^{\frac{2+\alpha}{N-2}\xi_{i,\varepsilon}}}{e^{\frac{2(2+\alpha)}{N-2}\xi_{i,\varepsilon}} + |x|^{2+\alpha}} \right)^{\frac{N-2}{2+\alpha}} + \pi_{\mu_{i,\varepsilon},\alpha}(x) \right) + |x|^{2-N}\varphi_\varepsilon(x),$$

for all $x \in \mathbb{R}^N \setminus B_1$, where $|x|^{2-N}\varphi_\varepsilon(x) \to 0$ uniformly on compacts contained in $\mathbb{R}^N \setminus B_1$ as $\varepsilon \to 0$ and $\mu_{i,\varepsilon} = e^{\frac{2}{N-2}\xi_{i,\varepsilon}} = M_i \varepsilon^{\frac{1-2i}{N-2}}$, is a solution of (1.4.5). This concludes the proof. □

Chapter 4

Conclusions and future problems

In this chapter we name some important conclusions obtained in this work. Firstly, Theorem (1.4.2) and Theorem (1.4.6) provide results on the existence of positive solutions for a slightly supercritical problem (and slightly subcritical for the exterior domain case), which are nontrivial extensions of the works by Liu in [27] and Del Pino, Dolbeault, and Musso in [13], where they considered the unit ball B_1 as the domain, whereas we chose a more general domain with certain symmetries. To be more specific, the results prove that, for supercritical exponents, it is possible to address the question of the existence of positive solutions to the problem (3.1.1) in some star-shaped domains, which is undoubtedly a remarkable fact given the well-known Pohozaev non-existence result for classical solutions. Moreover, this existence result also extends to the case of the exterior domain. The same can also be said for Theorem 1.4.1, but this result is even more remarkable because, as far as we know, no such result exists in the literature, making it a significant achievement and we believe it will be valuable for future works involving the Bilaplacian operator. Finally, another very interesting aspect of this work is that the solutions found were constructed explicitly, providing a very precise asymptotic profile of the actual solution to the problem.

We would also like to add that the present work is far from being a final study of the problems investigated. An interesting problem that could be analyzed is the existence of positive or sign-changing solutions for the Hénon problem

$$\begin{cases} \Delta^2 u = |x|^{\alpha} u^{p^*_{\alpha,2}} + \lambda_\varepsilon u & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases}$$

where $p^*_{\alpha,2} = \frac{N+4+2\alpha}{N-4}$ and $\lambda_\varepsilon \to 0$ as $\varepsilon \to 0$. According to what is presented in this thesis, a key point for finding solutions to the problem is to explicitly know the solutions to the limiting case, in other words, when the domain is $\mathbb{R}^N$. A recent result proved the existence of radially symmetric solutions with respect to the origin for the problem

$$\begin{cases} \Delta^2 u = |x|^{\alpha} u^{p^*_{\alpha,2}} & \text{in } \mathbb{R}^N, \\ u > 0 & \text{in } \mathbb{R}^N, \end{cases}$$

so, it would be interesting to see if this result can be expanded for a bounded domain.

Lastly, it seems very interesting to study the problem

$$\begin{cases} (-\Delta)^s u = & |u|^{p_s^*+\varepsilon} + \lambda_\varepsilon |x|^\beta u & \text{in } B_1, \\ u > & 0 & \text{in } B_1, \\ u = & 0 & \text{on } \mathbb{R}^N \backslash B_1, \end{cases}$$

where Δ^s is the fractional Laplacian operator, with $0 < s < 1$, $N > 2s$ and $p_s^* = \frac{N+2s}{N-2s}$. This operator has garnered significant interest in the field of partial differential equations in recent years, with new results being continuously obtained, making it very fascinating to study.

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